12/3/25

Conditioning on Random Variables

$$P(Y=y \mid X=x) = \frac{P(Y=y \text{ and } X=x)}{P(X=x)} \quad \leftarrow \neq 0$$

Two discrete random variables X and Y, we define a new random variable $Y \mid X=x$, where $x \in \text{Range}(X)$ and $P(X=x) \neq 0$, that has pmf

$$P_{Y|X=x}(y) = P(Y=y \mid X=x) = \frac{P(Y=y \text{ and } X=x)}{P(X=x)}$$

$$= \frac{P_{(Y,X)}(y,x)}{P_X(x)}$$

Example copied from previous class: - 20 birds in a circle (or n birds)

- at some point they all turn left or right w/ probability $\frac{1}{2}$

How did we model this?

$S = \{$all sequences of length 20 made from L or R$\}$

└ The standard product probability space

Let U be the number of unseen birds

Let U_i be 1 if bird i is unseen else 0

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what is the pmf of $P_{u_3|u_5=1}$?

$$P_{u_3|u_5=1}(1) = P(u_3=1 \mid u_5=1) = 0$$

(if bird 5 is unseen, bird 4 must be looking at bird 3)

$$P_{u_3|u_5=1}(0) = P(u_3=0 \mid u_5=1) = 1$$

what is the pmf of $P_{u_3|u_5=0}$?

$$P_{u_3|u_5=0}(1) = P(u_3=1 \mid u_5=0) = \frac{P(u_3=1 \text{ and } u_5=0)}{P(u_5=0)}$$

$$u_3 = 0 = \{\cdots L \underset{\text{idx } 3}{*} R \cdots\}$$

$$P(u_3=1 \text{ and } u_5=0) = P(u_3=1) = \frac{1}{4}$$

(bernoulli random variable)

$$P(u_5=0) = 1 - P(u_5=1) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$u_5 = 0 = \left\{\begin{array}{l} \cdots R \underset{\text{idx } 5}{*} R \cdots, \\ \cdots R * L \cdots, \\ \cdots L * L \cdots \end{array}\right\} \longrightarrow \text{or}\ldots \quad P(u_5=0) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow P_{u_3|u_5=0}(1) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$\Rightarrow P_{u_3|u_5=0}(0) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$E(u_3) = \frac{1}{4}$$

$$E(u_3 \mid u_5=1) = 0$$

$$E(u_3 \mid u_5=0) = \frac{1}{3}$$

What is the pmf of $P_{u_3|u_8=1}$?

$$P_{u_3|u_8=1}(1) = P(u_3=1 \mid u_8=1)$$

(independent)

$$= \frac{P(u_3=1 \text{ and } u_8=1)}{P(u_8=1)} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}$$

$$P_{u_3|u_8=1}(0) = P(u_3=0 \mid u_8=1) = 1 - \frac{1}{4} = \frac{3}{4}$$

What is the pmf of $P_{u_3|u_8=0}$?

independent

$$P_{u_3|u_8=0}(1) = \frac{P(u_3=1 \text{ and } u_8=0)}{P(u_8=0)} = \frac{\frac{1}{4}\cdot\frac{1}{4}}{\frac{1}{4}} = \frac{1}{4}$$

$$P_{u_3|u_8=1}(0) = P(u_3=0 \mid u_8=0) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$E(u_3 \mid u_8=1) = \frac{1}{4}$$

$$E(u_3 \mid u_8=0) = \frac{1}{4}$$

$$E(u_3) = \frac{1}{4}$$

Example: X and Y d.r.v., independent

$$X \sim \text{Binom}(m, p) \qquad Y \sim \text{Binom}(n, p)$$

$$P(X=j \mid X+Y=r) = \frac{P(X=j \text{ and } X+Y=r)}{P(X+Y=r)}$$

$$= \frac{P(X=j \text{ and } Y=r-j)}{P(X+Y=r)} = \frac{P(X=j)\,P(Y=r-j)}{P(X+Y=r)}$$

$$= \frac{\binom{m}{j} p^j (1-p)^{m-j} \binom{n}{r-j} p^{r-j} (1-p)^{n-r+j}}{\binom{m+n}{r} p^r (1-p)^{m+n-r}}$$

$$= \frac{\binom{m}{j}\binom{n}{r-j}}{\binom{m+n}{r}}$$

$$X \mid X+Y=r \sim \text{HGeom}(m, n, r)$$

Corollary 3.24: If X and Y are independent discrete r.v.,

then $$E(Y \mid X = x) = E(Y)$$

In general: $$E(Y \mid X = x) = \frac{1}{P_X(x)} \sum_{y \in Range(Y)} y \cdot P_{(Y,X)}(y, x)$$