12/1/25

Recall: 2 random variables X and Y on (S, P)

Consider $V = (X, Y)$

$$P_V : \text{Range}(X) \times \text{Range}(Y) \longrightarrow [0, 1]$$

$$P_V((x, y)) = P(X = x \text{ and } Y = y)$$

(recall)
Example: - 20 birds in a circle (or n birds)

- at some point they all turn left or right w/ probability $\frac{1}{2}$

How did we model this?

$S = \{$all sequences of length 20 made from L or R$\}$

The standard product probability space

Let U be the number of unseen birds

Let U_i be 1 if bird i is unseen else 0

ex: $U_3(LLLRRL\cdots) = 1$

$U_3(LLRLRL\cdots) = 0$

$\uparrow\uparrow\uparrow\uparrow\uparrow$
0 1 2 3 4

Let's look at U_3 and U_5:

define $g(x,y) = xy$; $g(U_3, U_5) = U_3 U_5$

$$\text{range}(U_3 U_5) = \{0, 1\}$$

$$E(g(U_3, U_5)) = E(U_3 U_5) = 0 \cdot P(U_3 U_5 = 0) + 1 \times P(U_3 U_5 = 1)$$

$$= P(U_3 U_5 = 1)$$

$$= P(U_3 = 1 \text{ and } U_5 = 1) = 0$$

always makes one of them seen!

L * L/R * R
$\uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow$
2 3 4 5 6

(i.e. $U_3 = 0$ or $U_5 = 0$ always)

<u>LOTUS</u>: $$E(g(X,Y)) = \sum_{\substack{x \in Range(X) \\ y \in Range(Y)}} g(x,y)\, P_{(X,Y)}(x,y)$$

we computed this above

Let $X = U_3$ $Y = U_5$

$g(X,Y) = XY$

$$= \sum_{\substack{x \in Range(U_3) \\ y \in Range(U_5)}} xy\, P_{(U_3,U_5)}(x,y)$$

$$= \sum_{\substack{x \in \{0,1\} \\ y \in \{0,1\}}} xy\, P_{(U_3,U_5)}(x,y)$$

only $x=1, y=1$ remains from sum

$$= 1 \cdot 1 \cdot P_{(U_3,U_5)}(1,1)$$

$$= 1 \cdot 1 \cdot P(U_3 = 1 \text{ and } U_5 = 1) = 0$$

<u>Def. 3.12 (covariance and correlation):</u>

X and Y discrete random variables

Let $V = (X,Y)$ $\quad cov(X,Y) = E((X-\mu_X)(Y-\mu_Y))$

$\mu_X = E(X)$, $\mu_Y = E(Y)$ $\quad = E(XY) - E(X)E(Y)$

The correlation coefficient of X, Y is:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}}$$

Two discrete random variables are called <u>uncorrelated</u> if:

$$\rho(X, Y) = 0 = \text{cov}(X, Y)$$

<u>Proposition 3.13:</u> (no linearity)

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab\,\text{cov}(X, Y)$$

<u>corollary 3.14:</u> If X and Y are uncorrelated, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

<u>Proposition 3.16:</u>

- X and Y are independent discrete random variables
- f and g are functions

Then, $f(X)$ and $g(Y)$ are independent and

$$E(f(X)g(Y)) = E(f(X))\,E(g(Y))$$

In particular, $\text{cov}(X, Y) = 0$ and $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

also, X and Y are uncorrelated

<u>Corollary 3.17:</u>

If $X_1, \ldots, X_n$ are independent discrete random variables, then

$$\text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n)$$

$X_i \sim \text{Bern}(p)$ $\quad E(X_i) = p$ $\quad Var(X_i) = p(1-p)$

$X \sim \text{Binom}(n, p)$ $\quad X = X_1 + \dots + X_n$

$$
\begin{aligned}
Var(X) &= Var(X_1) + \dots + Var(X_n) \\
&= p(1-p) + \dots + p(1-p) \\
&= np(1-p)
\end{aligned}
$$