

12/1/25

Recall: 2 random variables X and Y on (S, P)

Consider $V = (X, Y)$

$$P_V : \text{Range}(X) \times \text{Range}(Y) \rightarrow [0, 1]$$

$$P_V((x, y)) = P(X=x \text{ and } Y=y)$$

(recall)

Example: - 20 birds in a circle (or n birds)
- at some point they all turn left or right w/ probability $\frac{1}{2}$

How did we model this?

$$S = \{\text{all sequences of length 20 made from L or R}\}$$

↳ The standard product probability space

Let U be the number of unseen birds

Let U_i be 1 if bird i is unseen else 0

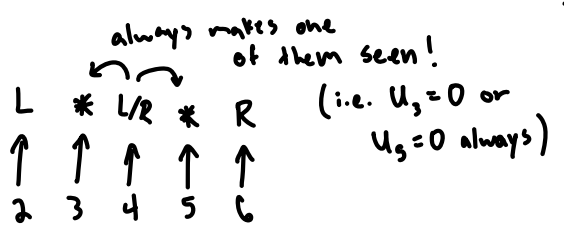
Let's look at U_3 and U_5 :

$$\begin{aligned} \text{ex: } U_3(LLLRRL \dots) &= 1 \\ U_3(LLRLRL \dots) &= 0 \\ &\quad \uparrow \uparrow \uparrow \uparrow \uparrow \\ &\quad 0 \ 1 \ 2 \ 3 \ 4 \end{aligned}$$

define $g(x, y) = xy$; $g(U_3, U_5) = U_3 U_5$

$$\text{range}(U_3 U_5) = \{0, 1\}$$

$$\begin{aligned} E(g(U_3, U_5)) &= E(U_3 U_5) = 0 \cdot P(U_3 U_5 = 0) + 1 \cdot P(U_3 U_5 = 1) \\ &= P(U_3 U_5 = 1) \\ &= P(U_3 = 1 \text{ and } U_5 = 1) = 0 \end{aligned}$$



LOTUS: $E(g(X, Y)) = \sum_{\substack{x \in \text{Range}(X) \\ y \in \text{Range}(Y)}} g(x, y) P_{(X, Y)}(x, y)$

Let $X = U_3$ $Y = U_5$

$g(x, y) = xy$

↑
we computed
this above

$= \sum_{\substack{x \in \text{Range}(U_3) \\ y \in \text{Range}(U_5)}} xy P_{(U_3, U_5)}(x, y)$

$= \sum_{\substack{x \in \{0, 1\} \\ y \in \{0, 1\}}} xy P_{(U_3, U_5)}(x, y)$

only $x=1, y=1$
remains from sum

$= 1 \cdot 1 \cdot P_{(U_3, U_5)}(1, 1)$

$= 1 \cdot 1 \cdot P(U_3 = 1 \text{ and } U_5 = 1) = 0$

Def. 3.12 (covariance and correlation):

X and Y discrete random variables

Let $V = (X, Y)$ $\text{COV}(X, Y) = E((X - \mu_X)(Y - \mu_Y))$

$\mu_X = E(X), \mu_Y = E(Y)$ $= E(XY) - E(X)E(Y)$

The correlation coefficient of X, Y is:

$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$

Two discrete random variables are called uncorrelated if:

$$\rho(X, Y) = 0 = \text{cov}(X, Y)$$

Proposition 3.13: (no linearity)

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{cov}(X, Y)$$

Corollary 3.14: If X and Y are uncorrelated, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

Proposition 3.16:

- X and Y are independent discrete random variables
- f and g are functions

Then, $f(X)$ and $g(Y)$ are independent and

$$E(f(X)g(Y)) = E(f(X))E(g(Y))$$

In particular, $\text{cov}(X, Y) = 0$ and $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

also, X and Y are uncorrelated

Corollary 3.17:

If $X_1, \dots, X_n$ are independent discrete random variables, then

$$\text{Var}(X_1 + \dots + X_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n)$$

$$X_i \sim \text{Bern}(p) \quad E(X_i) = p \quad \text{Var}(X_i) = p(1-p)$$

$$X \sim \text{Binom}(n, p) \quad X = X_1 + \dots + X_n$$

$$\text{Var}(X) = \text{Var}(X_1) + \dots + \text{Var}(X_n)$$

$$= p(1-p) + \dots + p(1-p)$$

$$= np(1-p)$$