11/21/25

* 20 birds in a circle facing the center
* At a moment they randomly turn left (L) or right (R)
* Number the birds 0, 1, 2, ..., 19

How do we model each bird with probability theory?

$$B_i = \{L, R\} \qquad P(\{L\}) = \frac{1}{2} \quad P(\{R\}) = \frac{1}{2}$$

How do we model all the birds?

$$S = \{\text{all sequences made from L and R of length 20}\}$$

$$= B_0 \times B_1 \times B_2 \times \cdots \times B_{19}$$

What is the probability function?

- we have 2^{20} such sequences
- we take the uniform probability distribution
- recall this is the unique probability function if the birds are independent

What is the event that bird 3 turns left?

$$E_3 = B_0 \times B_1 \times B_2 \times \{L\} \times B_4 \times \ldots \times B_{19}$$

$|E_3| = 2^{19}$ ↳ all sequences of the form $***L**\cdots*$

$$P(E_3) = \frac{|E_3|}{|S|} = \frac{2^{19}}{2^{20}} = \frac{1}{2}$$

i.e., $P(B_0 \times B_1 \times B_2 \times \{L\} \times B_4 \times \dots \times B_{19})$

$$= P(B_0) \cdot P(B_1) \cdot P(B_2) \cdot P(\{L\}) \cdot P(B_4) \cdot \dots \cdot P(B_{19})$$

$$= 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot \dots \cdot 1 = \frac{1}{2}$$

* A bird is unseen if no neighbor looks at it.

Take a sequence $s = s_0 s_1 s_2 \cdots s_{19}$.

Spot/bird i is unseen if $s_{i-1 \bmod 20} = L$ and $s_{i+1 \bmod 20} = R$

Ex: RLLLLL...LLL

$s_{19-1 \bmod 20} = s_{18} = L$

- bird 19 is the only unseen bird

$s_{19+1 \bmod 20} = s_0 = R$

Define $U_i : S \to \mathbb{R}$ such that

$$U_i(s) = \begin{cases} 1 & \text{if bird } i \text{ is unseen } (s_{i-1 \bmod 20} = L \text{ and } s_{i+1 \bmod 20} = R) \\ 0 & \text{if bird } i \text{ is seen} \end{cases}$$

Ex: $U_3(LLLLL \cdots L) = 0$

$U_{19}(RLLL \cdots LL) = 1$

$P(U_3 = 1) = \frac{1}{4}$ — spot 2 is L and spot 4 is R

$P(U_3 = 0) = \frac{3}{4}$ — spot 2 is L and spot 4 is L
spot 2 is R and spot 4 is R
spot 2 is R and spot 4 is L

$$E(U_3) = P(U_3 = 1) = \frac{1}{4}$$

$$E(U_i) = P(U_i = 1) = \frac{1}{4}$$

For a sequence $s = s_0 s_1 s_2 \cdots s_{19}$ in S we define the unseen support of s to be $USup(s)$, the collection of all unseen birds/spots. Then, let $U : S \to \mathbb{R}$ be defined by:

$$U(s) = |USup(s)|$$

- In English, U is the random variable that counts the number of unseen birds!

<u>Claim:</u> $U = U_0 + U_1 + U_2 + \cdots + U_{19}$

<u>Proof:</u> Let $s \in S$. [we need to show

$$U(s) = (U_0 + U_1 + \cdots + U_{19})(s)]$$

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$$U(s) = |USup(s)|$$

$$= \sum_{i=1}^{|USup(s)|} 1 = \sum_{i \in USup(s)} 1$$

$$= \sum_{i \in USup(s)} U_i(s) = \sum_{i \in USup(s)} U_i(s) + \sum_{j \notin USup(s)} U_j(s)$$

$$= U_0(s) + U_1(s) + \cdots + U_{19}(s)$$

$$= (U_0 + U_1 + \cdots + U_{19})(s) \quad \square$$

$$\Rightarrow E(U) = E(U_0 + U_1 + \cdots + U_{19})$$

$$= E(U_0) + E(U_1) + \cdots + E(U_{19})$$

$$= \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4} = \frac{20}{4} = 5$$

works for n birds too. (for $n \geq 3$)

$$E(U) = \frac{n}{4}$$

Ex: 3 birds

LLL	RLL
LLR	RLR
LRL	RRL
LRR	RRR

$$\frac{6}{8} = \frac{3}{4}$$