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(Chapter 3) Multivariate discrete distributions

Let X, Y be discrete random variables on a probability space (S, P). New function $(X, Y): S^2 \to \mathbb{R}^2$

$$(s_x, s_y) \to (X(s_x), Y(s_y))$$

random vector

Joint Probability Mass Functions

Let $V = (X, Y)$

$$P_V : \text{Range}(X) \times \text{Range}(Y) \longrightarrow [0,1]$$

$$P_V(x, y) = P(X = x \text{ and } Y = y) = P(X = x, Y = y)$$

Independent Random Variables

If X and Y are independent, then

$$P(X = x, Y = y) = P(X = x) P(Y = y)$$

Example Draw 2 cards from a standard deck of 52 cards. X is the number of hearts and Y is the number of queens

we know:
1. 13 hearts, 4 queens
2. 16 cards are either heart or queen
3. 36 cards are neither heart nor queen
4. 12 hearts are not queen
5. 3 queens are not hearts

★ Find the pmf of $V = (X, Y)$

$$\text{Range}(X) = \{0,1,2\}, \quad \text{Range}(Y) = \{0,1,2\}$$

$$P_V(2,2) = P(\text{2 hearts \& 2 queens}) = 0$$

$$P_V(0,0) = P(\text{0 hearts \& 0 queens}) = 0$$

$$P_V(1,0) = P(\text{1 heart \& 0 queens}) = \frac{12 \cdot 36}{\binom{52}{2}}$$

$$P_V(0,1) = \frac{3 \cdot 36}{\binom{52}{2}}$$

$$P_V(2,1) = \frac{12 \cdot 1}{\binom{52}{2}} \qquad P_V(2,0) = \frac{\binom{12}{2}}{\binom{52}{2}}$$

$$P_V(1,2) = \frac{1 \cdot 3}{\binom{52}{2}} \qquad P_V(0,2) = \frac{\binom{3}{2}}{\binom{52}{2}}$$

$$P_V(1,1) = \frac{12 \cdot 3 + 1 \cdot 36}{\binom{52}{2}}$$

Proposition 3.3: Suppose X and Y are d.r.v. then

$$P_X(x) = \sum_{y \in Range(Y)} P_{(X,Y)}(x,y)$$

$$P_Y(y) = \sum_{x \in Range(X)} P_{(X,Y)}(x,y)$$

Theorem 3.8 (LOTUS):

$$E(g(X,Y)) = \sum_{\substack{x \in Range(X) \\ y \in Range(Y)}} g(x,y)\, P_{(X,Y)}(x,y)$$

Corollary 3.9 (Linearity of E)

$$E(c_1 X_1 + \dots + c_n X_n) = c_1 E(X_1) + \dots + c_n E(X_n)$$

Example: 20 birds labeled $1, \dots, 20$ are sitting in a circle facing its center. Each bird turns left or right with probability $\frac{1}{2}$. Let U be the number of birds unseen by any neighbor.

$$E(U) = ?$$

Let B_i be the event that bird i is not seen by either neighbor. We want I_{B_i}, $P(B_i) = \frac{1}{4}$

$$E[I_{B_i}] = P(B_i) = \frac{1}{4}$$

$$\begin{aligned}
E[U] &= E[I_{B_1} + \dots + I_{B_{20}}] \\
&= E[I_{B_1}] + \dots + E[I_{B_{20}}] \\
&= \underbrace{\frac{1}{4} + \dots + \frac{1}{4}}_{20 \text{ times}} \\
&= \frac{20}{4} = 5
\end{aligned}$$