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<u>Theorem</u> (68 - 95 - 99.7% rule) :

If $X \sim N(\mu, \sigma^2)$ then

$$P(|X-\mu| < \sigma) \approx 0.68$$

$$P(|X-\mu| < 2\sigma) \approx 0.95$$

$$P(|X-\mu| < 3\sigma) \approx 0.997$$

<u>Gamma Distribution</u>:

The gamma distribution with parameters $v, \lambda > 0$ are defined by the probability density functions

$$g_v(x;\lambda) = \begin{cases} \frac{\lambda^v}{\Gamma(v)} x^{v-1} e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

check pdf:

$u = \lambda x$

$$\int_{-\infty}^{\infty} g_v(x;\lambda)\, dx = \int_0^{\infty} \frac{\lambda^v}{\Gamma(v)} x^{v-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} x^{v-1} e^{-\lambda x}\, dx = \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} \left(\frac{u}{\lambda}\right)^{v-1} e^{-u} \frac{1}{\lambda}\, du$$

$$= \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} \frac{1}{\lambda^v} u^{v-1} e^{-u}\, du = \frac{1}{\Gamma(v)} \int_0^{\infty} u^{v-1} e^{u}\, du$$

$$= \frac{1}{\Gamma(v)} \Gamma(v) = 1 \quad \checkmark$$

<u>Recall:</u> Fix a probability space (S, P)

Take random variables $X_1, X_2, \ldots, X_n : S \to \mathbb{R}$

1. $X_1, \ldots, X_n$ are independent if for any $x_1, \ldots, x_n$ in $\mathbb{R}$ we have:

$$P(X_1 \leq x_1 \text{ and } \cdots \text{ and } X_n \leq x_1) = P(X_1 \leq x_1) \cdots P(X_n \leq x_n)$$

2. $X_1, \ldots, X_n$ are identically distributed if they all have the same CDF.

3. We say that $X_1, \ldots, X_n$ are i.i.d. if they are independent and identically distributed.

<u>Theorem:</u> If $X_1, \ldots, X_n$ are exponential with λ and i.i.d., then

$$X_1 + X_2 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$$

(Busville / Exponential Random variable ; $X \sim \text{Gamma}(n, \lambda)$) each bus avg. wait time of $\frac{1}{\lambda}$ minutes

"what is the probability of waiting X minutes across the n buses I took this week?"

"what is the total time wasted in the collective consciousness of n people, each waiting for a different bus?"

X random variable

$g: \mathbb{R} \to \mathbb{R}$ $\quad g(X)$ is a random variable

Is $g(X)$ continuous? Not always.

Let $g(x) = -x$ $\quad Y = g(X)$

is $g(X)$ continuous?

$$P(g(X) \le y) = \int_{-\infty}^{y} p_{g(X)}(x)\,dx$$

$$F_Y(y) = P(Y \le y) = P(-X \le y) = P(X \ge -y)$$

$$= 1 - P(X < -y) = 1 - F_X(-y)$$

$$F'_Y(y) = 0 - F'_X(-y)(-1) = F'_X(-y)$$

$$\searrow \quad p_{g(X)}(x) = p_X(-x)$$

Let X be a random variable, $g(x) = ax + b$ $\quad a > 0$.

Let $Y = g(X)$.

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right)$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \Rightarrow F'_Y(y) = F'_X\left(\frac{y-b}{a}\right)\frac{1}{a} = p_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$