

11/12/25

Theorem (68 - 95 - 99.7% rule) :

If $X \sim N(\mu, \sigma^2)$ then $P(|X - \mu| < \sigma) \approx 0.68$

$$P(|X - \mu| < 2\sigma) \approx 0.95$$

$$P(|X - \mu| < 3\sigma) \approx 0.997$$

Gamma Distribution:

The gamma distribution with parameters $v, \lambda > 0$ are defined by the probability density functions

$$g_v(x; \lambda) = \begin{cases} \frac{\lambda^v}{\Gamma(v)} x^{v-1} e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

check pdf:

$$\int_{-\infty}^{\infty} g_v(x; \lambda) dx = \int_0^{\infty} \frac{\lambda^v}{\Gamma(v)} x^{v-1} e^{-\lambda x} dx$$

$$u = \lambda x$$

$$= \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} x^{v-1} e^{-\lambda x} dx = \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} \left(\frac{u}{\lambda}\right)^{v-1} e^{-u} \frac{1}{\lambda} du$$

$$= \frac{\lambda^v}{\Gamma(v)} \int_0^{\infty} \frac{1}{\lambda^v} u^{v-1} e^{-u} du = \frac{1}{\Gamma(v)} \int_0^{\infty} u^{v-1} e^{-u} du$$

$$= \frac{1}{\Gamma(v)} \Gamma(v) = 1 \quad \checkmark \quad \therefore$$

Recall: Fix a probability space (S, P)

Take random variables $X_1, X_2, \dots, X_n: S \rightarrow \mathbb{R}$

1. $X_1, \dots, X_n$ are independent if for any $x_1, \dots, x_n$ in $\mathbb{R}$ we have:

$$P(X_1 \leq x_1 \text{ and } \dots \text{ and } X_n \leq x_n) = P(X_1 \leq x_1) \cdots P(X_n \leq x_n)$$

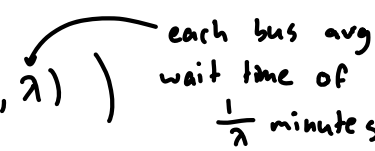
2. $X_1, \dots, X_n$ are identically distributed if they all have the same CDF.

3. We say that $X_1, \dots, X_n$ are i.i.d. if they are independent and identically distributed.

Theorem: If $X_1, \dots, X_n$ are exponential with

λ and i.i.d., then

$$X_1 + X_2 + \dots + X_n \sim \text{Gamma}(n, \lambda)$$

(Busville / Exponential Random variable ; $X \sim \text{Gamma}(n, \lambda)$)  each bus avg. wait time of $\frac{1}{\lambda}$ minutes

"What is the probability of waiting X minutes across the n buses I took this week?"

"What is the total time wasted in the collective consciousness of n people, each waiting for a different bus?"

X random variable

$g: \mathbb{R} \rightarrow \mathbb{R}$ $g(X)$ is a random variable

Is $g(X)$ continuous? Not always.

Let $g(x) = -x$ $Y = g(X)$

is $g(X)$ continuous?

$$P(g(X) \leq y) = \int_{-\infty}^y P_{g(X)}(x) dx$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(-X \leq y) = P(X \geq -y) \\ &= 1 - P(X < -y) = 1 - F_X(-y) \end{aligned}$$

$$F'_Y(y) = 0 - F'_X(-y)(-1) = F'_X(-y)$$

$$\downarrow P_{g(X)}(x) = P_X(-x)$$

Let X be a random variable, $g(x) = ax + b$ $a > 0$.

Let $Y = g(X)$.

$$F_Y(y) = P(Y \leq y) = P(ax + b \leq y) = P(X \leq \frac{y-b}{a})$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \Rightarrow F'_Y(y) = F'_X\left(\frac{y-b}{a}\right) \frac{1}{a} = P_X\left(\frac{y-b}{a}\right) \frac{1}{a}$$