11/10/25

Recall: Exponential Random Variable

$X \sim \text{Exp}(\lambda) \qquad P_X(t) = \begin{cases} 0 & t < 0 \\ \lambda e^{-\lambda t} & t \geq 0 \end{cases}$

$$F(t) = P(X \leq t) = 1 - e^{-\lambda t}$$

$$P(X > t) = e^{-\lambda t}$$

— —

Every δ seconds, we perform a Bernoulli trial with probability p_δ (of success)

we want to find the probability of the first success happening at the n-th "coin flip".

$\longrightarrow \quad A \sim \text{Geom}(p_\delta)$

Let T be the waiting time for the first success.

Let $t = n\delta$.

$$\hookrightarrow \quad P(T > t) = P(A > n)$$

$$= (1 - p_\delta)^n$$

$$= \left((1 - p_\delta)^{-\frac{1}{p_\delta}}\right)^{-\frac{p_\delta}{\delta} t}$$

Let $p_\delta = \lambda\delta$, then,

$$= \left((1 - \lambda\delta)^{-\frac{1}{\delta\lambda}}\right)^{-\lambda t}$$

what happens as $\delta \to 0^+$?

$$P(T > t) \to e^{-\lambda t}$$

$$X \sim \text{Exp}(\lambda) \quad P(X > t) = e^{-\lambda t}$$

Exponential is just a limit of Geometric

Recall: $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$

$$\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt$$

Let $t = x^2$. Then $dt = 2x\,dx$ and $t^{-\frac{1}{2}} = x^{-1}$

$$\int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt = \int_0^\infty 2e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx$$

$$= \int_0^\infty e^{-x^2}\, dx + \int_0^\infty e^{-x^2}\, dx$$

$$= \int_{-\infty}^{0} e^{-x^2}\, dx + \int_0^\infty e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

so far, $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx$

Let $x = \frac{s}{\sqrt{2}}$. Then $x^2 = \frac{s^2}{2}$ and $dx = \frac{1}{\sqrt{2}}\, ds$.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2}}\, ds = \sqrt{\pi}$$

$$1 = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\, ds}$$

it is a valid pdf :)

Normal Distribution:

We say that X is normally distributed or Gaussian and write $X \sim N(\mu, \sigma^2)$ if its pdf is

$$\gamma_{\mu,\sigma}(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

why do people care about the normal distribution?

The Central Limit Theorem.

Under certain conditions, if we add random variables, then their sum will behave like a normal random variable as the terms in the sum $\to \infty$

We say that X is standard normal if $X \sim N(0,1)$

$$\gamma_{0,1}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$X \sim N(\mu, \sigma^2)$

$$\int_{-\infty}^{\infty} \gamma_{\mu,\sigma}(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

let $u = \frac{y-\mu}{\sigma}$

$du = \frac{1}{\sigma} dy$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{u^2}{2}} \sigma \cdot du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{u^2}{2}} du = 1$$

(shown earlier)

The CDF of $X \sim N(0,1)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt$$

$X \sim N(\mu, \sigma^2)$

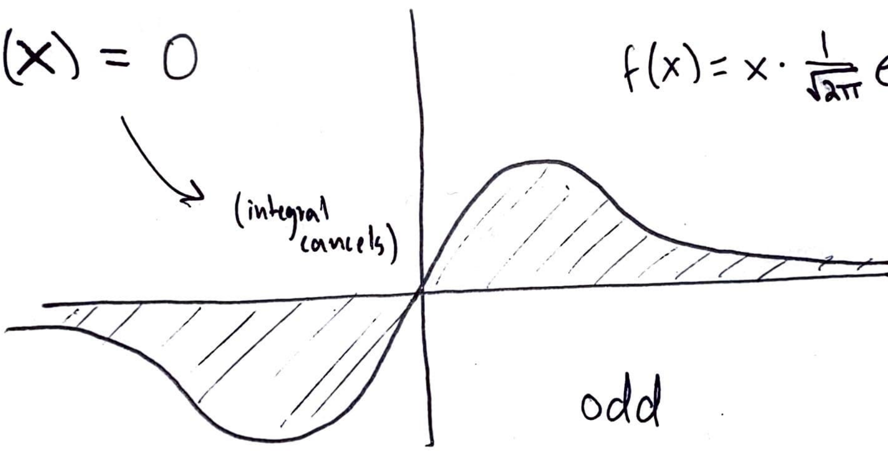
$E(X) = \mu \quad , \quad Var(X) = \sigma^2$

$X \sim N(0,1)$

Let k be odd.

$\mu_k(X) = 0$

$$f(x) = x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



for even moments:

$$\mu_{2n}(X) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$