11/7/25

Recall "Carville" (Memoryless Property)

$$P(T > t_0 + t \mid T > t_0) = P(T > t)$$

Theorem: There are unique random variables with the memoryless property

proof: (not doing it; involves solving a differential equation)

Exponential Random Variable:

* Generalization of geometric distribution!

$$X \sim Exp(\lambda), \quad \lambda > 0$$

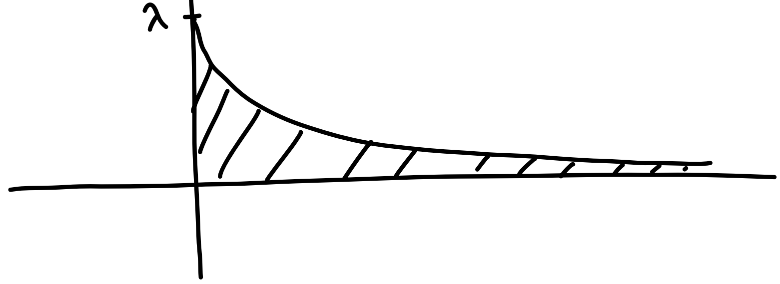
$$P_X(t) = \begin{cases} 0, & t > 0 \\ \lambda e^{-\lambda t}, & t \geq 0 \end{cases}$$

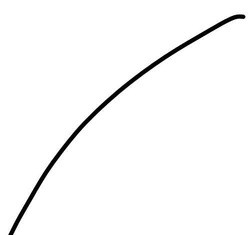
Note that if $t = 0$, $P_X(t)$ could be > 1. This is fine for a pdf. (not for pmf)

because we <u>integrate</u>

Is $P_X(t)$ a valid pdf?

$$\int_{-\infty}^{\infty} P_X(t) = 1$$





(assume this)

$$\int_{-\infty}^{\infty} p_X(t) = \int_0^{\infty} \lambda e^{-\lambda t}\,dt = \lim_{a\to\infty} \int_0^a \lambda e^{-\lambda t}\,dt$$

$$= \lim_{a\to\infty} -e^{-\lambda t}\Big|_0^a = \lim_{a\to\infty} \left(-e^{-\lambda a} + e^{-\lambda\cdot 0}\right)$$

$$= \lim_{a\to\infty} \left(1 - \frac{1}{e^{\lambda a}}\right) = \underline{1}$$

what about the CDF of X ?

$$F_X(t) = P(X \le t) = \int_0^t p_X(t)\,dt = \int_0^t \lambda e^{-\lambda t}\,dt$$

$$= 1 - e^{-\lambda t}$$

we want $P(X > t_0 + t \mid X > t_0) = P(X > t)$

$$\text{LHS} = \frac{P(X > t_0 + t \text{ and } X > t_0)}{P(X > t_0)} = \frac{P(X > t_0 + t)}{P(X > t_0)}$$

$$= \frac{1 - P(X \le t_0 + t)}{1 - P(X \le t_0)} = \frac{1 - (1 - e^{-\lambda t_0 - \lambda t})}{1 - (1 - e^{-\lambda t_0})} = \frac{e^{-\lambda t_0 - \lambda t}}{e^{-\lambda t_0}} = e^{-\lambda t}$$

$$\text{RHS} = 1 - P(X \le t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$

☆ This is the only pdf that satisfies memoryless principle

$$E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

In carville,

- Buses have no schedule
- Average wait time is 10 minutes

How do we model this?

$$X \sim Exp\left(\frac{1}{10}\right) \qquad \left(E(x) = 10 = \frac{1}{\frac{1}{10}} \right)$$

— —

The Gamma function is the function $\Gamma : (0,\infty) \to \mathbb{R}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

The Beta function is a function on two positive variables $x, y > 0$ such that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Proposition 2.70: The following hold.

1. $\Gamma(1) = 1$

2. $\Gamma(x+1) = x\Gamma(x), \quad \forall x > 0$

   * Continuous analog of factorials

3. For any $n \in \{1,2,3,4,\ldots\}$, $\Gamma(n) = (n-1)!$

4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

5. For any $x, y > 0$ we have

$$\int_0^1 s^{x-1}(1-s)^{y-1}\,ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y)$$

— —

$X \sim Exp(\lambda)$

$x = \lambda t \quad dx = \lambda\,dt$

$t = \frac{x}{\lambda} \quad t^k = x^k \lambda^{-k}$

$$\mu_k(X) = \int_0^\infty t^k \lambda e^{-\lambda t}\,dt$$

$$= \int_0^\infty x^k \lambda^{-k} e^{-x}\,dx$$

$$= \lambda^{-k}\int_0^\infty x^k e^{-x}\,dx$$

$$= \lambda^{-k}\Gamma(k+1)$$

$$= \lambda^{-k} \cdot k!$$

$\Rightarrow E(X) = \frac{1}{\lambda}$ (first moment)

$\Rightarrow Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$