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Corollary 2.62: we have

1. $E(aX+b) = aE(X)+b$
2. $\mu_k(X) = E(X^k)$
3. $Var(X) = E((X-\mu)^2) = E(X^2) - (E(X))^2$
4. $Var(aX+b) = a^2 Var(X)$

Corollary 2.63. $f, g: \mathbb{R} \to \mathbb{R}$, $f(x) \leq g(x)$ for all $x \in \mathbb{R}$

Then, $E(f(X)) \leq E(g(X))$

Proposition 2.64 (Markov Inequality):

Suppose Y is an integrable, nonnegative random variable.

Then for any $c > 0$ we have $P(Y > c) \leq \frac{1}{c} E(Y)$

Ex: let Y be the income of a person

$c = 2E(Y)$

$P(Y > c) = \frac{1}{c} E(Y) = \frac{1}{2E(Y)} E(Y) = \frac{1}{2}$

Theorem 2.65 (Chebyshev's Inequality):

Suppose X is a 2-integrable random variable.

Let $\mu = E(X)$, $v = E((X-\mu)^2) = Var(X)$ and

$\sigma = \sqrt{v}$ the standard deviation.

Then for any $c, r > 0$ we have:

$$P(|X - \mu| \geq c\sigma) \leq \frac{1}{c^2}$$

used all the time with probabilistic method

$$P(|X - \mu| > r) \leq \frac{\sigma^2}{r^2} = \frac{Var(X)}{r^2}$$

Ex: $X \sim Binom(20, \frac{1}{2})$

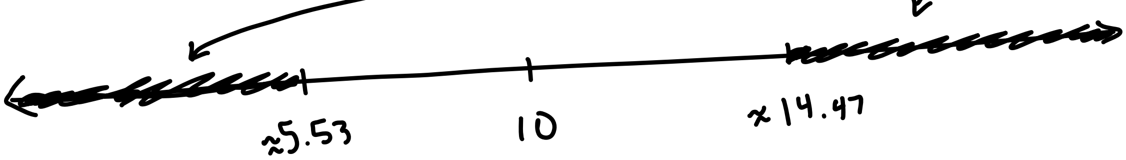
$$\mu = np = 20 \cdot \tfrac{1}{2} = 10$$

$$v = n \cdot p(1-p) = 5$$

$$\sigma = \sqrt{5}$$

$$P(|X - 10| \geq \underbrace{\overset{c}{2}\sqrt{5}}_{\approx 4.4721}) \leq \frac{1}{2^2} = \frac{1}{4}$$

probability of getting this many heads is $\leq \frac{1}{4}$



Alex lives in Busville. All the buses in Busville are always exactly on time and operate 24/7. The time between buses is 10 minutes. Alex does not have a watch ☹. Alex shows up at the bus stop at a uniformly random time.

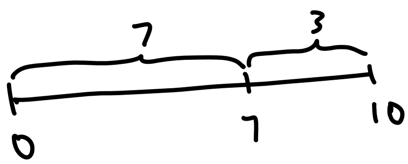
- What is the distribution of how long Alex has to wait for the bus?

  $\text{Unif}(0, 10)$

- What is the average time that Alex waits?

  5 minutes $\left(E[X] = \frac{10+0}{2} = 5 \right)$

- If Alex just showed up, what is the probability of him waiting at least 3 minutes?

  $\frac{7}{10}$

- Given that Alex already waited 6 minutes, what is the probability of him waiting at least 3 more?

  Let T be the total waiting time

  $$P(T \geq 6+3 \mid T \geq 6) = \frac{P(T \geq 9 \text{ and } T \geq 6)}{P(T \geq 6)}$$

Alex moves to carville. In carville, many people have cars and they don't take the bus. Alex does not have a car :(. The average wait time is 10 minutes, but there is no schedule!

- This means:

$$P(T \geq 6+3 \mid T \geq 6) = P(T \geq 3)$$

- and more generally, we want:

$$P(T \geq t_0 + t \mid T \geq t_0) = P(T \geq t)$$

what is a distribution that models this?

$\Rightarrow$ <u>Exponential Distribution:</u> (this is the unique distribution)

$T \sim Exp(\lambda)$ if its pdf is $P_T(t) = \begin{cases} 0, & t < 0 \\ \lambda e^{-\lambda t}, & t \geq 0 \end{cases}$