11/3/25

Continuous Random Variables:

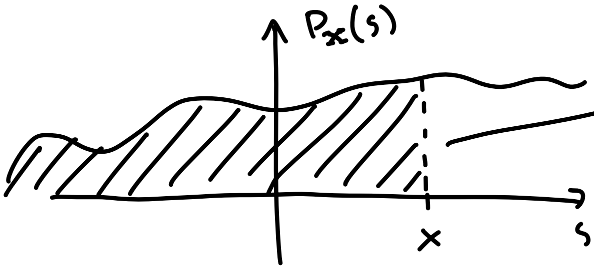
How do we model picking random numbers in the interval $[0,1]$?

A random variable X is continuous if there exists a function $p_X : \mathbb{R} \to [0,\infty)$ such that

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} p_X(s)\,ds \quad , \text{ for all } x \in \mathbb{R}$$

cumulative distribution function (cdf)

How do you define $\int_{-\infty}^{x} p_X(s)\,ds$?



Area under the curve (Not trivial)

"Your book doesn't go into it, most books don't. I am not doing it in class."

You want:

* $1 = P(X < \infty) = \int_{-\infty}^{\infty} p_X(s)\,ds$

* $F_X'(x) = p_X(x)$, for all but countably many x

* $P(X < c) = P(X \le c) = F_X(c)$

$*$ $P(a \le X \le b) = P(x \le b) - P(x \le a)$

$$= F_X(b) - F_X(a) = \int_a^b p_X(s)\, ds$$

$*$ $P(X = c) = 0$

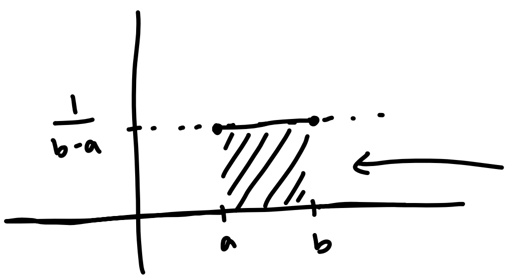
<u>Proposition 2.56:</u> Any function $p : \mathbb{R} \to [0, \infty)$ satisfying:

$\int_{-\infty}^{\infty} p(x)\,dx = 1$ is the pdf of some random variable.

$X \sim \mathrm{Unif}(a,b)$ if $\quad p_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b] \end{cases}$

$$\int_{-\infty}^{\infty} p_X(s)\,ds = \int_a^b p_X(s)\,ds = \int_a^b \frac{1}{b-a}\,ds = \left.\frac{s}{b-a}\right|_a^b$$

$$= \frac{b}{b-a} - \frac{a}{b-a} = \frac{b-a}{b-a} = 1$$



area is 1 !!

— —

Take an interval $[c,d] \subseteq [a,b]$

$$P(c \le X \le d) = \int_c^d p_X(s)\,ds = \int_c^d \frac{1}{b-a}\,ds = \left.\frac{s}{b-a}\right|_c^d$$

$$= \frac{c}{b-a} - \frac{d}{b-a} = \frac{c-d}{b-a}$$ ← matches our intuition!

$E(X) = ?$ picking from interval $[a,b]$

intuition (correct) : $E(X) = \frac{a+b}{2}$

formally: expectation is the exact same as in discrete case, just swap Σ with a $\int$:)

$$E(X) = \int_{-\infty}^{\infty} x \, P_X(x) \, dx$$

Suppose X is a continuous random variable with pdf

$P_X : \mathbb{R} \to [0, \infty)$ and $s \geq 1$ is a real number.

We say that X is s-integrable if

$$\int_{-\infty}^{\infty} |x|^s P_X(x) \, dx < \infty$$

And if X is s-integrable we define

$$\mu_s(X) = \int_{-\infty}^{\infty} x^s P_X(x) \, dx$$

We say that X is integrable if X is 1-integrable and define

$$E(X) = \int_{-\infty}^{\infty} x \, P_X(x) \, dx$$

Returning to our example over $[a,b]$:

$$X \sim \text{Unif}(a,b), \qquad P_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b] \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \, P_X(x)\, dx = \int_a^b x\, P_X(x)\, dx = \int_a^b x \, \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{b^2}{2} \cdot \frac{1}{b-a} - \frac{a^2}{2} \cdot \frac{1}{b-a} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$$Var(X) = \int_{\mathbb{R}} (x-\mu)^2 P_X(x)\, dx$$

$$\int_{-\infty}^{\infty}$$

$$\sigma(X) = \sqrt{Var(X)}$$

<u>Proposition 2.59</u>: $Var(X) = \mu_2(X) - \mu_1(X)^2$

<u>Proposition 2.60</u>: If X is nonnegative then

$$E(X) = \int_0^{\infty} P(X > x)\, dx$$

Theorem 2.61: (LOTUS)

$$E(f(X)) = \int_{\mathbb{R}} f(x) P_X(x)\, dx$$

- Linearity of Expectation holds for continuous case as well