10/31/25

How to improve in problem solving?

* Think about a problem for a long time
  - 5 minutes is too short
  - 10 minutes is too short
  - 30 minutes is acceptable before seeking help

* when thinking about a problem,
  - ask questions (what does this word mean?)
  - read the problem a bunch of times

* when asking for help,
  - BAD: Someone tells you the problem, you write it up and never think about it again
    - → You should instead leave the problem and try it again in a few days
  - ideally, get help from a person (in-person)

From last class:

We have $1. We flip a coin. Each time it lands H, double money. Each time it lands T, game stops and I give you all the money.

How much should you pay to play?

Should you bet all your money on this?

Let X be the random variable of the coin flipping

$X \sim Geom\left(\frac{1}{2}\right)$

$$E\left(2^{X-1}\right) = \infty$$

- You will eventually have a lot of money if you play this game, but it will take a long time to get there. Longer than your lifetime

☆ Expectation should not be the only thing you consider when talking about games like this.

Proof of LOTUS:

<u>Claim:</u>

X is a discrete random variable and $f: \mathbb{R} \to \mathbb{R}$ is a function, then

$$E(f(X)) = \sum_{x \in range(X)} f(x) P_X(x)$$

<u>Proof:</u> One has,

$$E(f(X)) = \sum_{y \in Range(f(X))} y \cdot P_{f(X)}(y) \quad \text{(Definition of Expectation)}$$

$$= \sum_{y \in Range(f(X))} y \cdot P(f(X) = y) \quad \text{(By definition of pmf)}$$

$$= \sum_{y \in \mathrm{Range}(f(X))} y \cdot P(\{s \in S \mid f(\underbrace{X(s)}_{x}) = y\}) \quad \text{(By definition of } P(X=x))$$

$$= \sum_{y \in \mathrm{Range}(f(X))} y \cdot P\Big(\bigcup_{\substack{x \in \mathrm{Range}(X) \\ f(x) = y}} \{s \in S \mid X(s) = x\} \Big)$$

each s can only map to one $x \in \mathrm{Range}(X)$, otherwise X is not a function

(since X is a function, all sets that make up the union are disjoint)

$$= \sum_{y \in \mathrm{Range}(f(X))} y \cdot \sum_{\substack{x \in \mathrm{Range}(X) \\ f(x) = y}} P(\{s \in S \mid X(s) = x\}) \quad \text{(By additivity)}$$

$$= \sum_{y \in \mathrm{Range}(f(X))} y \cdot \sum_{\substack{x \in \mathrm{Range}(X) \\ f(x) = y}} P(X = x)$$

$$= \sum_{y \in \mathrm{Range}(f(X))} y \cdot \sum_{\substack{x \in \mathrm{Range}(X) \\ f(x) = y}} p_X(x)$$

(Distributivity)

$$= \sum_{y \in \mathrm{Range}(f(X))} \sum_{\substack{x \in \mathrm{Range}(X) \\ f(x) = y}} y \cdot p_X(x)$$

$$= \sum_{x \in \mathrm{Range}(X)} f(x) \cdot p_X(x) \quad \text{(By Fubini-Tonelli)}$$

Corollary 2.50 (Monotonicity of Expectation)

Suppose X is a d.r.v. and $f, g : \text{Range}(X) \to \mathbb{R}$ are functions such that $f(x) \leq g(x)$ for all $x \in \text{Range}(X)$

Then, $E(f(X)) \leq E(g(X))$

Proof:

One has,

$$E(f(X)) \overset{\text{LOTUS}}{=} \sum_{x \in \text{Range}(X)} f(X) P_X(x) \overset{f(x) \leq g(x)}{\leq} \sum_{x \in \text{Range}(X)} g(X) P_X(x) \overset{\text{LOTUS}}{=} E(g(X))$$

Corollary 2.49

X is a discrete random variable with range $\{0, 1, 2, \dots\}$. Then, $G_X(s) = E(s^X)$