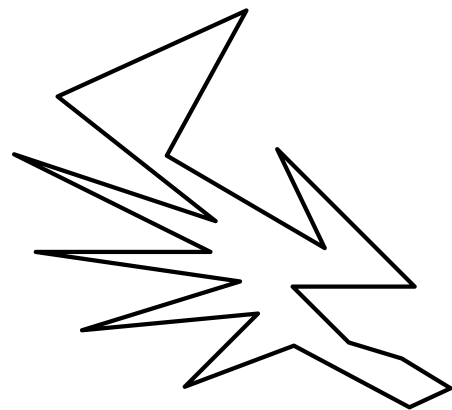


10/31/25

How to improve in problem solving?

* Think about a problem for a long time

- 5 minutes is too short
- 10 minutes is too short
- 30 minutes is acceptable before seeking help



* When thinking about a problem,

- ask questions (what does this word mean?)
- read the problem a bunch of times

* When asking for help,

- BAD: Someone tells you the problem, you write it up and never think about it again
 - You should instead leave the problem and try it again in a few days
- ideally, get help from a person (in-person)

From last class:

We have \$1. We flip a coin. Each time it lands H, double money.

Each time it lands T, game stops and I give you all the money.

How much should you pay to play?

Should you bet all your money on this?

Let X be the random variable of the coin flipping

$$X \sim \text{Geom}\left(\frac{1}{2}\right)$$

$$E(2^{X-1}) = \infty$$

- You will eventually have a lot of money if you play this game, but it will take a long time to get there. Longer than your lifetime

★ Expectation should not be the only thing you consider when talking about games like this.

Proof of LOTUS:

Claim:

X is a discrete random variable and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function, then

$$E(f(X)) = \sum_{x \in \text{range}(X)} f(x) P_X(x)$$

Proof: One has,

$$E(f(X)) = \sum_{y \in \text{Range}(f(X))} y \cdot P_{f(X)}(y) \quad (\text{Definition of Expectation})$$

$$= \sum_{y \in \text{Range}(f(X))} y \cdot P(f(X) = y) \quad (\text{By definition of pmf})$$

$$= \sum_{y \in \text{Range}(f(X))} y \cdot P(\{s \in S \mid \underbrace{f(X(s))}_{x} = y\}) \quad (\text{By definition of } P(X=x))$$

$$= \sum_{\substack{y \in \text{Range}(f(X)) \\ f(x)=y}} y \cdot P\left(\bigcup_{\substack{x \in \text{Range}(X) \\ f(x)=y}} \{s \in S \mid X(s) = x\}\right)$$

each s can only map to one $x \in \text{Range}(X)$, otherwise X is not a function

(since X is a function, all sets that make up the union are disjoint)

$$= \sum_{y \in \text{Range}(f(X))} y \cdot \sum_{\substack{x \in \text{Range}(X) \\ f(x)=y}} P(\{s \in S \mid X(s) = x\}) \quad (\text{By additivity})$$

$$= \sum_{y \in \text{Range}(f(X))} y \cdot \sum_{\substack{x \in \text{Range}(X) \\ f(x)=y}} P(X=x)$$

$$= \sum_{y \in \text{Range}(f(X))} y \cdot \sum_{\substack{x \in \text{Range}(X) \\ f(x)=y}} P_X(x)$$

(Distributivity)

$$= \sum_{y \in \text{Range}(f(X))} \sum_{\substack{x \in \text{Range}(X) \\ f(x)=y}} y \cdot P_X(x)$$

$$= \sum_{x \in \text{Range}(X)} f(x) \cdot P_X(x) \quad (\text{By Fubini-Tonelli})$$

Corollary 2.50 (Monotonicity of Expectation)

Suppose X is a d.r.v. and $f, g: \text{Range}(X) \rightarrow \mathbb{R}$ are functions such that $f(x) \leq g(x)$ for all $x \in \text{Range}(X)$

Then, $E(f(X)) \leq E(g(X))$

Proof:

One has,

$$E(f(X)) \stackrel{\text{LOTUS}}{=} \sum_{x \in \text{Range}(X)} f(x) P_X(x) \stackrel{f(x) \leq g(x)}{\leq} \sum_{x \in \text{Range}(X)} g(x) P_X(x) \stackrel{\text{LOTUS}}{=} E(g(X))$$

Corollary 2.49

X is a discrete random variable with

range $\{0, 1, 2, \dots\}$. Then, $G_X(s) = E(s^X)$