10/29/25

Recall:

LOTUS: $E(f(X)) = \sum_{i=1}^{\infty} f(x_i)\, p_X(x_i)$ Range X is $\{x_1, x_2, x_3, \ldots\}$

Linearity of Expectation: $E(f(X) + g(X)) = E(f(X)) + E(g(X))$

$$E(aX + b) = aE(X) + b$$

Example: Flip a coin 3 times. Let X be the number of heads obtained. You pay me \$1 to play. I give you the number of heads, squared, in dollars back.

$$E(X) = 0 \cdot \tfrac{1}{8} + 1 \cdot \tfrac{3}{8} + 2 \cdot \tfrac{3}{8} + 3 \cdot \tfrac{1}{8} = \tfrac{3}{2}$$

TTT

HTT
THT
TTH

HHT
HTH
THH

HHH

or...
$X \sim \text{Binom}(3, \tfrac{1}{2})$
$E(X) = 3 \cdot \tfrac{1}{2} = \tfrac{3}{2}$

How much will I pay you on average if you play the game?
(LOTUS, $f(x) = x^2$)

$$E(X^2) = 0^2 \cdot \tfrac{1}{8} + 1^2 \cdot \tfrac{3}{8} + 2^2 \cdot \tfrac{3}{8} + 3^2 \cdot \tfrac{1}{8} = 3$$

How much will you win on average after

$$E(X^2 - 1) = E(X^2) - 1 = 3 - 1 = 2$$

Linearity of Expectation

should you play the game? Yes, people like money

The point: in general, $E(f(X)) \neq f(E(X))$

Should you play if I ask for \$10 to play?

$$E(X^2 - 10) = E(X^2) - 10 = 3 - 10 = -7$$

No, people like positive money.

— —

I have \$1. Flip a coin until we get tails.

Each time we get heads, double the money.

When you get tails, I pay you all the money.

What is the maximum amount of money for which you should be willing to play?

Let X be the spot of the first tail T.

$X \sim \text{Geom}(\frac{1}{2})$

$E(X) = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$ ← this isn't useful.

Goal:

$E(2^{X-1}) = ?$

$$P(X=n) = \underbrace{\left(\tfrac{1}{2}\right)^{n-1}}_{\text{heads}}\underbrace{\left(\tfrac{1}{2}\right)}_{\text{tail}} = \left(\tfrac{1}{2}\right)^n$$

$$E(X) = \sum_{n=1}^{\infty} n\left(\tfrac{1}{2}\right)^n \longrightarrow$$

$$E(2^{X-1}) = E(2^X \cdot \tfrac{1}{2})$$
$$= \tfrac{1}{2} E(2^X)$$
$$= \tfrac{1}{2}\sum_{n=1}^{\infty} 2^n\left(\tfrac{1}{2}\right)^n$$
$$= \tfrac{1}{2}\sum_{n=1}^{\infty} 1 = \infty$$

But 2^x is not integrable!

- In this, it is fine! The sum is all positive integers!

$\Rightarrow$ So, expectation suggests that we should put everything we have on this game! But nobody should do this.

Imagine you have $\$2^{32000}$ (~unlimited money), and the game costs $\$2^{32}$ to play:

probability	net winnings	result
$\frac{1}{2}$	$-2^{32}+1$	T
$\frac{1}{4}$	$-2^{32}+2$	HT
$\frac{1}{8}$	$-2^{32}+2^2$	HHT
$\frac{1}{2^{32}}$	$-2^{32}+2^{31}=2^{31}(-2+1)$ $=-2^{31}$	$\underbrace{HH\cdots H}_{31}T$
just to break even! $\frac{1}{2^{33}}$	$-2^{32}+2^{32}=0$	$\underbrace{HH\cdots H}_{32}T$
though eventually, unfathomable amounts of money $\frac{1}{2^{300}}$	$-2^{32}+2^{299}$ $=2^{32}(2^{277}-1)$	$\underbrace{HH\cdots H}_{299}T$