10/27/25

Recall (it's been a while):

A random variable is a function $X : S \to \mathbb{R}$ where (S, P) is a probability space

A discrete random variable is a random variable that has range $\{x_1, x_2, x_3, \dots\} \subseteq \mathbb{R}$

The probability mass function of X is

$$P_X : \{x_1, x_2, x_3, \dots\} \to [0,1] \quad \text{such that} \quad P_X(x_i) = P(X = x_i)$$

$$P_X(x_i) = P(X = x_i) = P(\{s \in S \mid X(s) = x_i\})$$

<end of last class:>

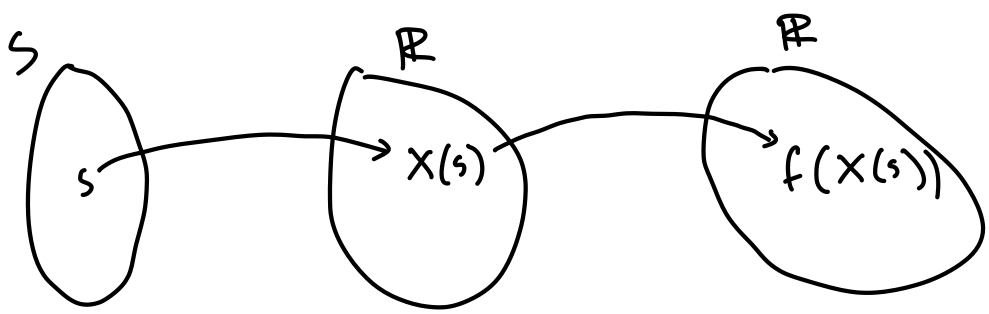
X is a random variable, $f : \mathbb{R} \to \mathbb{R}$ is a function

What is $f(X)$?

X is a function! There is some probability space (S, P) and we have $X : S \to \mathbb{R}$

$f(X)$ is a composition of functions

$$f(X) = f(X(s)) = f \circ X$$



what about $E(f(x))$?

A discrete random variable X with range $\{x_1, x_2, x_3, \ldots\}$ is s-integrable if:

$$\sum_{i=1}^{\infty} |x_i|^s P_X(x_i) = |x_1|^s P_X(x_1) + |x_2|^s P_X(x_2) + |x_3|^s P_X(x_3) + \ldots < \infty$$

If X is n-integrable, then we define the n-th moment of X to be:

$$\mu_n(X) = \sum_{i=1}^{\infty} x_i^n P_X(x_i)$$

Expectation is just the first moment of X:

$$E(X) = \sum_{i=1}^{\infty} x_i P(X = x_i) = \sum_{i=1}^{\infty} x_i^1 P_X(x_i) = \mu_1(X)$$

— — (end of recap) — —

What about $E(f(X))$? $\quad f(X) = X^2$

X has outputs: $0, 1, 2, 3, 4, 5, \ldots$

pmf values: $p_0, p_1, p_2, p_3, p_4, p_5, \ldots$

X^2 has outputs: $0, 1^2, 2^2, 3^2, 4^2, 5^2$

$$E(X) = \sum_{i=0}^{\infty} i \cdot P_X(i)$$

these are equivalent !!

$$E(f(X)) = E(X^2) = \sum_{i=0}^{\infty} i^2 \cdot P_{X^2}(i) = \sum_{i=0}^{\infty} i^2 \cdot P_X(i)$$

(LOTUS)

Theorem 2.46 (The law of the subconcious/unconscious statistician):

Suppose X is a discrete random variable with range $\{x_1, x_2, x_3, \dots\}$ and $P_X(x_k) = P(X = x_k)$. Then for any function $f: \mathbb{R} \to \mathbb{R}$, if $f(x)$ is integrable, then $E(f(x)) = \sum_{k=1}^{\infty} f(x_k) P_X(x_k)$

Corollary 2.47 (Linearity of Expectation):

Suppose X is a discrete random variable with range $\{x_1, x_2, x_3, \dots\}$, and take functions $f, g: \mathbb{R} \to \mathbb{R}$. Then,

$$E(f(x) + g(x)) = E(f(x)) + E(g(x))$$

(f, g, and $f+g$ are integrable)

In particular, $E(aX + b) = aE(X) + b$

Proof: let $h = f+g$. One has:

$$E(f(X) + g(X)) = E((f+g)(X)) = E(h(X))$$

(LOTUS)

$$= \sum_{k=1}^{\infty} h(x_k) P_X(x_k) = \sum_{k=1}^{\infty} (f+g)(X) P_X(X_k) = \sum_{k=1}^{\infty} (f(X) + g(X)) P_X(X_k)$$

$$= \sum_{k=1}^{\infty} f(x_k) P_X(x_k) + \sum_{k=1}^{\infty} g(x_k) P(x_k) = E(f(X)) + E(g(X))$$

(LOTUS)

$\square$