10/15/25

Proposition (2.39): Suppose X is a discrete random variable with range contained in $\{0, 1, 2, \dots\}$. If $G_X(s)$ is the probability generating function of X, then:

$$G_X(1) = 1 \qquad \mu_2[X] = G_X''(1) + G_X'(1)$$

$$E(X) = G_X'(1) \qquad Var[X] = G''_X(1) + G_X'(1) - (G_X'(1))^2$$

proof:

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \dots$$

$$p_i = P(X = i)$$

$$G_X(1) = p_0 + p_1 + p_2 + p_3 + \dots = 1$$

convergent power series, we take derivative of each term

$$G_X'(s) = p_1 + 2p_2 s + 3p_3 s^2 + 4p_4 s^3 + \dots = \sum_{n=0}^{\infty} n p_n s^{n-1}$$

$$G_X'(1) = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + 4 \cdot p_4 + \dots = E(X) \quad \left(\text{by definition of expectation}\right)$$

$\square$

$$X \sim Bin(n, p) \qquad E(X) = ?$$

binomial theorem

$$G_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k = (ps + (1-p))^n$$

$$G_X'(s) = n(ps + (1-p))^{n-1}(p)$$

$$G_X'(1) = np(p + 1 - p)^0 = np = E(X)$$

or... $X \sim Bin(n,p)$ $\quad x_1, x_2, \dots, x_n \sim Ber(p)$

$X = x_1 + x_2 + \dots + x_n$ $\quad$ ☆ Linearity of Expectation

$$E(X) = E(x_1 + x_2 + \dots + x_n) = E(x_1) + \dots + E(x_2)$$

—— ——

$X \sim Geom(p)$ $\quad E(x) = ?$

$$G_x(s) = \frac{ps}{1-(1-p)s}$$

$$G_x'(s) = \frac{p(1-(1-p)s) - ps(p-1)}{(1-(1-p)s)^2}$$

$$G_x'(1) = \frac{p(1-(1-p)) - p(p-1)}{(1-(1-p))^2} = \frac{p(p) - p^2 + p}{p^2} = \frac{p}{p^2}$$

$$G_x'(1) = \frac{1}{p}$$

—— ——

$X \sim NegBin(k,p)$ $\quad$ (using linearity of expectation, could differentiate if we wanted to)

$x_1, \dots, x_k \sim Geom(p)$

$X = x_1 + \dots + x_k$

$$E(X) = E(x_1 + \dots + x_k) = \overbrace{E(x_1) + \dots + E(x_k)}^{\frac{1}{p} + \dots + \frac{1}{p}} = \frac{k}{p}$$

$X \sim Poi(\lambda)$ $\quad E(x) = ?$ $\quad Var(X) = ?$

$$G_x(s) = e^{\lambda(s-1)}$$

$$G'_x(s) = \lambda e^{\lambda(s-1)}$$

$$G'_x(1) = \lambda e^{\lambda(1-1)} = \lambda e^0 = \lambda = E(x)$$

$$G''_x(s) = \lambda^2 e^{\lambda(s-1)} \qquad G''_x(1) = \lambda^2$$

$$Var(x) = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

X is a random variable.

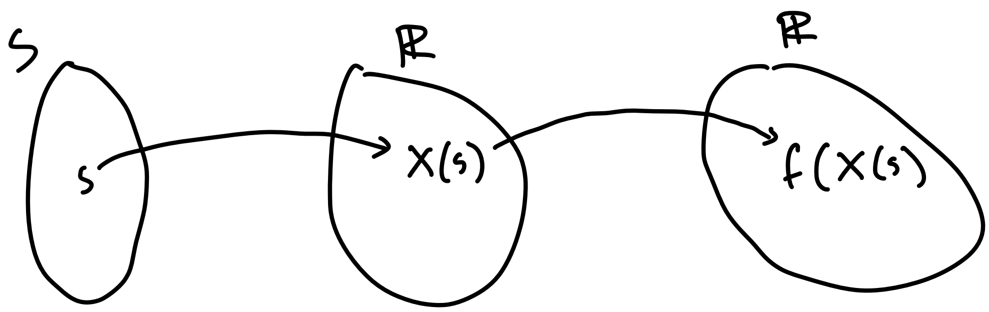
$f: \mathbb{R} \to \mathbb{R}$ is a function

What is $f(X)$?

X is a function! There is some probability space (S, P) and we have $X: S \to \mathbb{R}$

$f(X)$ is a composition of functions

$$f(X) = f(X(s)) = f \circ X$$



$f(X)$ is another random variable

examples: $f(X) = 100X$

$f(X) = X^2$