10/13/25

Roll a standard die. What is the number you will see on average?

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

$$= \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6$$

probabilities

outputs of a random variable

Let X be the random variable that records the number on the die. Suppose X is a discrete random variable with range $\{x_1, x_2, \dots\} \subseteq \mathbb{R}$ and a probability mass function p.

Let $s \in [1, \infty)$. We say that X is s-integrable and we write $X \in L^s$, if $\sum_{n=0}^{\infty} |x_n|^s p(x_n) < \infty$

(L^p in analysis books)

ex: $3.5 = \sum_{n=0}^{\infty} x_n p(x_n) = \sum_{n=1}^{6} |x_n|^1 p(x_n)$ $\quad (s=1)$

We say that X is integrable if it is 1-integrable and we define the mean/expectation/average value of X to be

$$E(x) = \sum_{n=0}^{\infty} x_n \, p(x_n)$$

If $n \in \{1, 2, 3, \dots\}$ and X is n-integrable then we define its n-th moment as

$$\mu_n[X] = \sum_{k=0}^{\infty} x_k^{\,n} \, p(x_k)$$

use of absolute value before relates to conditional convergence (calc II!)

- The 1st moment of X is its expected value

X discrete random variable with mass function p. If X is 2-integrable and its mean is $\mu = \mu_1[X]$, then we define variance to be

$$Var[x] = \sum_{n=0}^{\infty} (x_n - \mu)^2 \, p(x_n)$$

and the standard deviation

$$\sigma(x) = \sqrt{Var[x]}$$

$X \sim Ber(p)$

$$E[x] = 0 \cdot (1-p) + 1 \cdot p$$

outputs of X $\{0,1\}$

$p_x(x_n)$

Proposition (2.37): Suppose that X is integrable, discrete with range contained in $\{0,1,2,\ldots\}$. Then

$$E[X] = \sum_{n=0}^{\infty} P(X>n)$$

proof: from definition,

$$E[x] = \sum_{n=0}^{\infty} n\,P(X=n)$$

$$\begin{aligned}
\sum_{n=0}^{\infty} P(X>n) = \quad & P(X>0) = \overbrace{P(X=1)}^{1 \text{ time}} + P(X=2) + P(X=3) + \cdots \\
& \qquad\qquad\qquad\qquad 2 \text{ times} \\
+\; & P(X>1) = P(X=2) + P(X=3) + \cdots \\
& \qquad\qquad\qquad\qquad 3 \text{ times} \\
+\; & P(X>2) = P(X=3) + \cdots \\
+\; & \cdots \\
= \; & 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) + \cdots \\
= \; & \sum_{n=0}^{\infty} n\cdot P(X=n)
\end{aligned}$$

— —

Suppose X is the number of rolls of a die until you get a 6.

$$E[x] = \sum_{n=0}^{\infty} n\,P(X=n) = \sum_{n=0}^{\infty} P(X>n) = \sum_{n=0}^{\infty} \left(\tfrac{5}{6}\right)^n = \frac{1}{1-\frac{5}{6}} = \frac{1}{\frac{1}{6}} = 6$$

probability it takes more than n heads to get a 6

probability that first n heads are not 6