

10/13/25

Roll a standard die. What is the number you will see on average?

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

$$= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6$$

↑ ↑ ↑ ↑
probabilities outputs of
a random variable

Let X be the random variable that records the number on the die. Suppose X is a discrete random variable with range $\{x_1, x_2, \dots\} \subseteq \mathbb{R}$ and a probability mass function p .

Let $s \in [1, \infty)$. We say that X is s -integrable

and we write $X \in L^s$, if $\sum_{n=0}^{\infty} |x_n|^s p(x_n) < \infty$
(L^p in analysis books)

$$\text{ex: } 3.5 = \sum_{n=0}^{\infty} x_n p(x_n) = \sum_{n=1}^6 |x_n|^s p(x_n) \quad (s=1)$$

we say that X is integrable if it is 1-integrable and

we define the mean/expectation/average value of X to be

$$E(x) = \sum_{n=0}^{\infty} x_n p(x_n)$$

If $n \in \{1, 2, 3, \dots\}$ and X is n -integrable then we

define its n -th moment as

$$\mu_n[X] = \sum_{k=0}^{\infty} x_k^n p(x_k)$$

use of absolute value before relates
to conditional convergence (calc II!)

- The 1st moment of X is its expected value

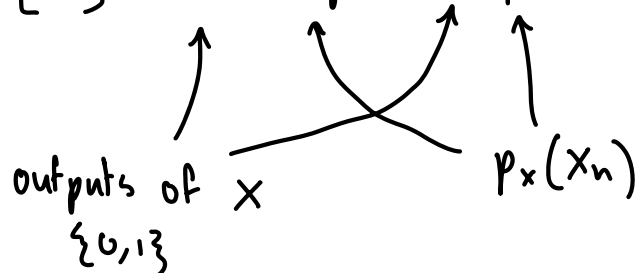
X discrete random variable with mass function p . If X is 2-integrable and its mean is $\mu = \mu_1[X]$, then

we define variance to be $\text{Var}[X] = \sum_{n=0}^{\infty} (x_n - \mu)^2 p(x_n)$

and the standard deviation $\sigma(x) = \sqrt{\text{Var}[x]}$

$$X \sim \text{Ber}(p)$$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p$$



Proposition (2.37): Suppose that X is integrable, discrete with range contained in $\{0, 1, 2, \dots\}$. Then

$$E[X] = \sum_{n=0}^{\infty} P(X > n)$$

proof: from definition,

$$E[X] = \sum_{n=0}^{\infty} n P(X = n)$$

$$\begin{aligned} \sum_{n=0}^{\infty} P(X > n) &= P(X > 0) = \overbrace{P(X=1) + P(X=2) + P(X=3) + \dots}^{1 \text{ time}} \\ &+ P(X > 1) = \overbrace{P(X=2) + P(X=3) + \dots}^{2 \text{ times}} \\ &+ P(X > 2) = \overbrace{P(X=3) + \dots}^{3 \text{ times}} \\ &+ \dots \\ &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \dots \\ &= \sum_{n=0}^{\infty} n \cdot P(X=n) \end{aligned}$$

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Suppose X is the number of rolls of a die until you get a 6.

$$E[X] = \sum_{n=0}^{\infty} n P(X = n) = \sum_{n=0}^{\infty} \underbrace{P(X > n)}_{\substack{\text{probability it takes} \\ \text{more than } n \text{ heads} \\ \text{to get a 6}}} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{5}{6}\right)^n}_{\substack{\text{probability that} \\ \text{first } n \text{ heads} \\ \text{are not 6}}} = \frac{1}{1 - \frac{5}{6}} = \frac{1}{\frac{1}{6}} = 6$$