10/10/25

"Math is either Linear Algebra, Calculus, or Combinatorics"

<u>Probability Generating Functions</u> (talking about probability using calculus)

Let $X$ be a random variable that is discrete and whose range is contained in $\{0, 1, 2, 3, \dots\}$

For $n \in \{0, 1, 2, 3, \dots\}$ define $p_n = P(X = n)$.

The probability generating function (pgf) of $X$ is the function:

$G_X : [0,1] \to \mathbb{R}$ $\qquad G_X(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \dots$

$$= \sum_{n=0}^{\infty} p_n s^n$$

$G(1) = p_0 + p_1 + p_2 + p_3 + \dots = 1$

For $s \in [0,1]$, $G(s)$ is (between 0 and 1, increasing and bounded by 1) a convergent series.

— —

$G_{Ber(p)}(s) = (1-p) + ps = g + ps$

$g = 1 - p$

shorthand variable

$X \sim Ber(p)$

$P_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$

$$G_{Bin(n,p)} = \binom{n}{0}p^0(1-p)^n + \binom{n}{1}p^1(1-p)^{n-1}s + \binom{n}{2}p^2(1-p)^{n-2}s^2$$

$$+ \binom{n}{3}p^3(1-p)^{n-3}s^3 + \dots + \binom{n}{n}p^n(1-p)^0 s^n + 0 + 0 + \dots$$

$$= \left( sp + (1-p) \right)^n \quad \text{(Binomial Theorem)}$$

$X \sim Bin(n,p)$

$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

— —

$$G_{Geom(p)}(s) = 0 + ps + (1-p)ps^2 + (1-p)^2 ps^3 + \dots + (1-p)^{n-1}ps^n + \dots$$

$$= ps \cdot \sum_{n=0}^{\infty} ((1-p)s)^n = ps\left(\frac{1}{1-(1-p)s}\right)$$

$$= \frac{ps}{1-(1-p)s} \quad \text{(Geometric Series)}$$

$X \sim Geom(p)$

$P_X(k) = (1-p)^{k-1}p$

— —

$$G_{NegBin(k,p)}(s) = \sum_{N=k}^{\infty} \binom{N-1}{k-1} p^k (1-p)^{N-k} s^N = (ps)^k \sum_{N=k}^{\infty} \binom{N-1}{k-1}(1-p)^{N-k}s^{N-k}$$

let $n = N-k$

$$= (ps)^k \sum_{n=0}^{\infty} \binom{n+k-1}{k-1}(1-p)^n s^n$$

using the known identity $\sum_{n=0}^{\infty} \binom{n+r-1}{r-1} t^n = (1-t)^{-r}$ :

$$= (ps)^k \left(1-(1-p)s\right)^{-k}$$

$$= \left(\frac{ps}{1-(1-p)s}\right)^k = \left(G_{Geom(p)}(s)\right)^k$$

$X \sim NegBin(k,p)$

$P_X(N) = \binom{N-1}{k-1} p^k (1-p)^{N-k}$

— —

$$G_{HGeom(w,b,n)}(s) = \frac{\binom{w}{0}\binom{b}{n}}{\binom{w+b}{n}} + \frac{\binom{w}{1}\binom{b}{n-1}}{\binom{w+b}{n}}s + \dots + \frac{\binom{w}{w}\binom{b}{n-w}}{\binom{w+b}{n}}s^w$$

$$= \sum_{k=0}^{w} \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}} s^k$$

$X \sim HGeom(w,b,n)$

$P_X(k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$

$$G_{Poi(\lambda)}(s) = e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!}s + e^{-\lambda}\frac{\lambda^2}{2!}s^2 + e^{-\lambda}\frac{\lambda^3}{3!}s^3 + \cdots$$

$X \sim Poi(\lambda)$

$P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$$= e^{-\lambda}\left(\frac{(\lambda s)^0}{0!} + \frac{(\lambda s)^1}{1!} + \frac{(\lambda s)^2}{2!} + \frac{(\lambda s)^3}{3!} + \cdots\right)$$

$$= e^{-\lambda}\left(e^{\lambda s}\right) \quad \text{(Taylor Series)}$$

$$= e^{\lambda s - \lambda} = e^{\lambda(s-1)}$$