10/3/25

Last class:

$X \sim Ber(p) \longrightarrow P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$

$X \sim Bin(p) \longrightarrow P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$X \sim Geom(p) \longrightarrow P_X(k) = (1-p)^{k-1} p$

$X \sim NegBin(k,p) \longrightarrow P_x(N) = \binom{N-1}{k-1} p^k (1-p)^{N-k}$

If $X \sim Geom(p)$, then $X \sim NegBin(1,p)$

The hypergeometric distribution

We have a bin containing w white balls and b black balls. We select n balls at random and denote by X the number of white balls selected.

$X \sim HGeom(w, b, n)$

possible outcomes of X: $0, 1, 2, 3, \dots, n$

$P_x(k) = P(X=k) = P(\text{picking } k \text{ white balls})$

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what is the sample space?

All subsets of balls of size n : $\binom{w+b}{n}$

→ possible choices of white balls $\binom{w}{k}$

→ possible choices of black balls $\binom{b}{n-k}$

$$\Rightarrow \quad P_X(k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$$

Discrete uniform distribution:

we say that a random variable X with outcomes $x_1, \ldots, x_n$ has the discrete uniform distribution if

$$P_X(x_i) = \frac{1}{n}$$

– some books write: $X \sim \text{Unif}(n)$

$X \sim \text{DUnif}(n)$

The Poisson distribution

A Poisson random variable with parameter $\lambda > 0$ is a discrete random variable X with range $\{0, 1, 2, \ldots\}$ and probability mass function $P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$

$$X \sim \text{Poi}(\lambda)$$

Can such an X exist?

we know: $P_X(0) + P_X(1) + P_X(2) + \dots = 1$

$$= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} + \dots$$

$$= e^{-\lambda}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \dots\right)$$

Taylor expansion for e^x

$$= e^{-\lambda} e^{\lambda} = 1$$

Relationship between Binomial and Poisson:

$X \sim Bin(n, p)$ and $Y \sim Poi(\lambda)$

Let $p = \frac{\lambda}{n}$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{(n)_k}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^k$$

what happens as $n \to \infty$?

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \quad (\text{calc II !})$$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^k = 1$$

$$\Longrightarrow \lim_{n\to\infty} P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} = P_Y(k)$$