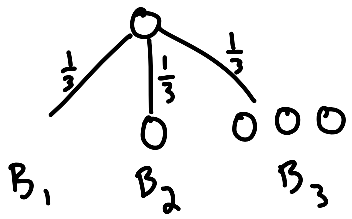
9/29/25

<u>recap of problem 2 on HW4:</u>



Let D be the event that all bacteria die eventually. $P(D) = ?$

$$P(D) = \frac{1}{3} + \frac{1}{3}P(D) + \frac{1}{3}P(D)^3$$

$$P(D) = 1 \quad \text{or} \quad P(D) = \frac{\sqrt{5}-1}{2}.$$ which one?

Let D_n be the probability that all bacteria die within n steps.

$$P(D_1) = \frac{1}{3}$$

(law of total probability)

$$P(D_n) = P(D_n|B_1)P(B_1) + P(D_n|B_2)P(B_2) + P(D_n|B_3)P(B_3)$$

$$= 1 \cdot \frac{1}{3} + P(D_{n-1}) \cdot \frac{1}{3} + P(D_{n-1})^3 \cdot \frac{1}{3}$$

$$P(D_n) = \frac{1}{3} + \frac{1}{3}P(D_{n-1}) + \frac{1}{3}P(D_{n-1})^3$$

<u>Claim:</u> $P(D_n) \leq \frac{\sqrt{5}-1}{2}$

<u>Proof:</u> By induction on n. (base case) If $n=1$, then $P(D_1) = \frac{1}{3} < \frac{\sqrt{5}-1}{2}$ (≈ 0.618)

(inductive hypothesis) Suppose $n \geq 2$ and the claim holds for all $n' < n$.

Then, $P(D_n) = \frac{1}{3} + \frac{1}{3}P(D_{n-1}) + \frac{1}{3}P(D_{n-1})^3 \leq \frac{1}{3} + \frac{1}{3}\left(\frac{\sqrt{5}-1}{2}\right) + \frac{1}{3}\left(\frac{\sqrt{5}-1}{2}\right)^3$

$$= \frac{\sqrt{5}-1}{2}$$

$$P(D_n) \leq \frac{\sqrt{5}-1}{2} \qquad \square$$

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Now, $P(D) = P\left(\bigcup_{n=1}^{\infty} D_n\right)$ $\qquad D_1 \subset D_2 \subset D_3 \subset \cdots \subset D$

(continuity from below) $= \lim_{n\to\infty} P(D_n) \leq \lim_{n\to\infty} \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}-1}{2}$

So, $P(D) \leq \frac{\sqrt{5}-1}{2}$ and $\left(P(D) = \frac{\sqrt{5}-1}{2} \text{ or } P(D) = 1 \right)$

$\Longrightarrow P(D) = \frac{\sqrt{5}-1}{2}$

<u>Last time</u>: Random Variables!

probability space (S, P), Function $X : S \to \mathbb{R}$

- We say that X is discrete if its range is a finite or countable set of real numbers.

<u>Definition (Probability Mass Function)</u>:

Given a random variable X, its probability mass function is

P_X : range of $X \to [0,1]$ such that

range of $X = \{x_1, x_2, x_3, \ldots\}$ $\qquad P_X(x_i) = P(X = x_i)$

<u>Bernoulli Random Variables</u>:

Fix a probability space (S, P), and take some event $E \subseteq S$ such that $P(E) = p \in [0,1]$

Define a function:

$$I_E : S \to \{0, 1\}$$

$$I_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases}$$

I_E is a random variable

I_E is called the indicator function with respect to E

I_E is called a Bernoulli random variable

$$P_{I_E}(1) = P(I_E = 1) = P(\{s \in S \mid \overset{s \in E}{I_E(s) = 1}\}) = P(E) = p$$

$$P_{I_E}(0) = P(I_E = 0) = P(\{s \in S \mid \overset{s \notin E}{I_E(s) = 0}\}) = P(E^c) = 1 - p$$

Binomial Random Variable:

Suppose we do n independent Bernoulli trials, each with probability p of success. Let X be the number of times that each Bernoulli trial works, that is, we get 1.

$$P_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

from sequence of 1's and 0's of length n $(x_1 x_2 x_3 \cdots x_n)$, we choose k of these to be 1. There are $\binom{n}{k}$ ways we can choose from the sequence

probability of our sequence having k 1's to begin with

probability of $n-k$ 0's occuring (we want <u>exactly</u> k 1's)