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Random Variables

A function $X : S \to \mathbb{R}$, where (S, P) is a probability space, is called a random variable.

Examples:

a) Roll a pair of dice. Let X be the sum of the numbers on the dice. Then X is a random variable.

$$X : \{1,2,\dots,6\} \times \{1,2,\dots,6\} \to \mathbb{R}, \quad X((a,b)) = a+b$$

$$S = \{1,2,\dots,6\} \times \{1,2,\dots,6\}$$

(S, P) is a probability space where we do uniform probability

$\uparrow$ set $\uparrow$ probability function

$\hookrightarrow P(E) = \dfrac{|E|}{|S|}$

b) Roll a die 1000 times. Let X be the number of times we get a 6. Then X is a random variable, with possible outputs $\{0, 1, 2, \dots, 1000\}$. $S = \{1,2,3,4,5,6\}$

Rolling 1000 times: $\overbrace{S \times S \times S \times \dots S}^{1000 \text{ times}} = S^{1000}$

$\to$ Unique probability function P on S^{1000}

$X : S^{1000} \to \mathbb{R}, \quad X((a_1, a_2, \dots, a_{999}, a_{1000})) = \# a_i == 6$

<u>Convention:</u> Random variables are denoted with capital letters

Take X example (a) from above:

$$P(X=2) = \frac{1}{36}$$

by definition

$$P(\{(a,b)\in S : X((a,b))=2\}) = P(\{(1,1)\})$$

$$P(X\leq 3) = P(\{(a,b)\in S : X((a,b)) \leq 3\}) = P(\{(1,1),(1,2),(2,1)\})$$

$$= \frac{3}{36} = \frac{1}{12}$$

Cumulative Distribution Function

Let X be a random variable. The cumulative distribution function (cdf) is the function $F : \mathbb{R} \to [0,1]$,

$$F(x) = P(X \leq x)$$

<u>Definition (Independence):</u> Fix a sample space (S,P).

The random variables $X_1,\dots,X_n : S \to \mathbb{R}$ are called <u>independent</u>, if for any numbers $x_1,\dots,x_n \in (-\infty,\infty]$, the events

$\{X_1 \leq x_1\},\dots,\{X_n \leq x_n\}$ are independent

Equivalently, $P(X_1 \le x_1, \dots, X_n \le x_n) = P(X_1 \le x_1) \cdots P(X_n \le x_n)$

($\cap$ and)

<u>Remark</u>: Random variables $X_1, \dots, X_n$ are independent iff for any set $B_1, \dots, B_n$ the events $\{B_1 \in X_1\}, \dots, \{B_n \in X_n\}$ are independent.