9/24/25

Last time:

Definition (Ramsey Number):

For $k, m \in \mathbb{N}$, $R(k,m)$ is the smallest number n such that if I invite n people to a party, k people know each other, or m people don't know each other.

↳ we showed: $R(3,3) = 6$

Theorem (Ramsey 1929): $R(k,m)$ is finite.

Theorem (Erdős): If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$

Thus, $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$

proof: Construct a probability space, by coloring each line independently with probability $\frac{1}{2}$ for both red and blue.

Let S be a subset of $\{1, 2, \ldots, n\}$ of size k, and A_S be the event that all the lines between the k vertices in S have the same color.

$$P(A_S) = 2 \cdot \left(\tfrac{1}{2}\right)^{\binom{k}{2}}$$

the number of total lines

There are two possible colors

Let A be the event that some subset of size k of $\{1, 2, \ldots, n\}$ is monochromatic (all lines same color)

$$P(A) = \underbrace{\binom{n}{k}}_{\substack{\#\text{ of subsets} \\ \text{of } n \text{ of size } k}} \underbrace{P(A_s)}_{\substack{\text{probability that} \\ \text{each subset (of size } k) \\ \text{is monochromatic}}} = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

(by initial condition)

$P(A) < 1$. Then $P(A^c) > 0$ because $P(A^c) = 1 - P(A)$.

So, there exists some coloring of the lines with no subsets of size k that are monochromatic.

Hence, $R(k,k) > n$.

Now, need to show $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$.

Let $n = \lfloor 2^{\frac{k}{2}} \rfloor \implies$ need to show $\binom{\lfloor 2^{k/2} \rfloor}{k} 2^{1-\binom{k}{2}} < 1$

(if this holds, we can substitute n for $\lfloor 2^{k/2} \rfloor$)

$$\binom{\lfloor 2^{\frac{k}{2}} \rfloor}{k} 2^{1-\binom{k}{2}} = \frac{(\lfloor 2^{\frac{k}{2}} \rfloor)_k}{k!} 2^{1-\binom{k}{2}} \leq \frac{(2^{\frac{k}{2}})_k}{k!} 2^{1-\binom{k}{2}}$$

because: $(x)_k = (x)(x-1)(x-2) \cdots (x-k+1)$

$(\lfloor x \rfloor)_k = \lfloor x \rfloor (\lfloor x \rfloor - 1)(\lfloor x \rfloor - 2) \cdots (\lfloor x \rfloor - k + 1)$

$(\lfloor x \rfloor)_k \leq (x)_k$

$$\binom{\lfloor 2^{\frac{k}{2}} \rfloor}{k} 2^{1-\binom{k}{2}} \le \frac{\left(2^{\frac{k}{2}}\right)^{\underline{k}}}{k!} 2^{1-\binom{k}{2}} = \frac{2^{\frac{k^2}{2}}}{k!} 2^{1-\frac{k(k-1)}{2}} = \frac{2^{\frac{k^2}{2}}}{k!} 2^{1-\frac{k^2}{2}+\frac{k}{2}}$$

always greater than a falling factorial

$$= \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad \text{for} \quad k \ge 3 .$$

Therefore, $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$ for $k \ge 3$

$\square$

check: $R(3,3) > \lfloor 2^{\frac{3}{2}} \rfloor$

$6 > 2$ ✓