9/19/25

Theorem (Baye's Rule):

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

proof: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (definition of $P(A|B)$)

$$= \frac{P(A \cap B)P(A)}{P(B)P(A)} \qquad \left(\frac{P(A)}{P(A)} = 1 \right)$$

$$= \frac{P(B \cap A) \cdot P(A)}{P(A) \cdot P(B)} \qquad (A \cap B = B \cap A)$$

$$= \frac{P(B|A)P(A)}{P(B)} \qquad \square$$

Ex: 1 fair coin

1 biased coin, $\frac{3}{4}$ probability of heads

1. pick a coin at random
2. flip the coin 3 times.

If the coin lands heads 3 times, what is the probability that the fair coin was chosen?

Let A = event fair coin chosen (A^c is biased coin)

Let B = event of heads 3x

problem asking: what is $P(A|B)$?

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{(bayes' rule)}$$

$$= \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \quad \text{(law of total probability)}$$

$$= \frac{\left(\frac{1}{2}\right)^3 \frac{1}{2}}{\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + \left(\frac{3}{4}\right)^3 \cdot \frac{1}{2}} \approx 0.23$$

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Bob tests positive for blue foot disease.

This disease affects 1% of the population.

The test is 95% accurate.

What is the probability that Bob has blue foot disease?

Let D be the event that Bob has blue foot

Let T be the event that Bob tests positive

☆ The test being 95% accurate means:

1. $P(T|D) = 0.95$
2. $P(T|D^c) = 0.05$

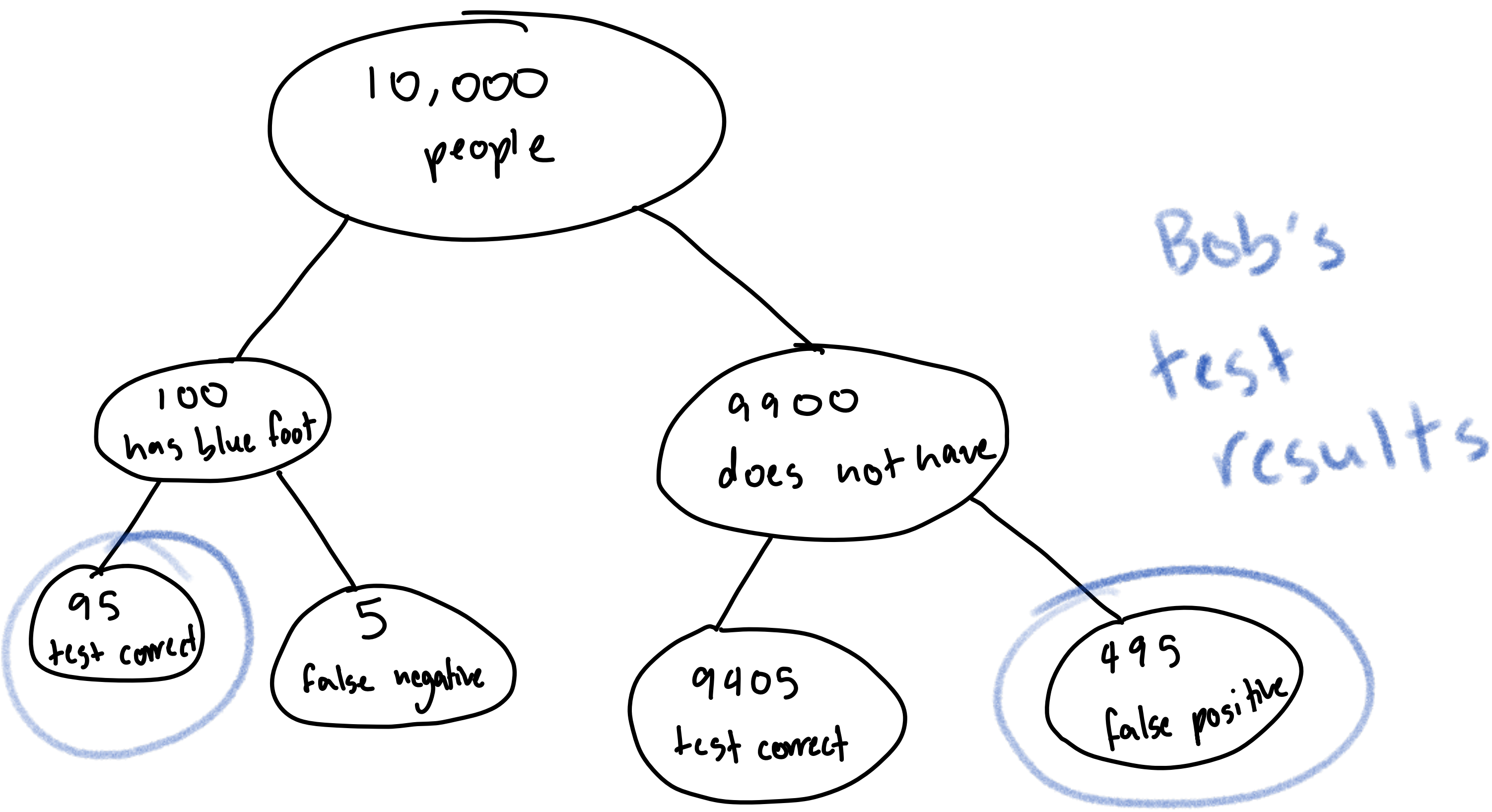
$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

$$= \frac{P(T|D)\,P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

$$= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.05 \cdot 0.99}$$

$$\approx 0.16$$

Why does this happen?



$$\frac{95}{95+495} \approx 0.16$$