9/15/25

(*)

Prop: If A and B are independent events then so are A and B^c

Proof:

$$\begin{aligned} P(A \cap B^c) &= P(A \setminus (A \cap B)) && (A \cap B^c = A \setminus (A \cap B)) \\ &= P(A) - P(A \cap B) && (\text{proposition } 1.7) \\ &= P(A) - P(A)P(B) && (A \text{ and } B \text{ are independent}) \\ &= P(A)(1 - P(B)) && (\text{algebra}) \\ &= P(A)P(B^c) && (\text{proposition } 1.7) \end{aligned}$$

Prop: If A and B are independent events then so are A^c and B^c.

well, first: (**) **Prop:** If A and B are independent events then so are A^c and B.

Proof: If A and B are independent, B and A are also independent. By (*), then B and A^c are independent. Therefore, A^c and B are independent.

Proof: By (*) A and B^c are independent

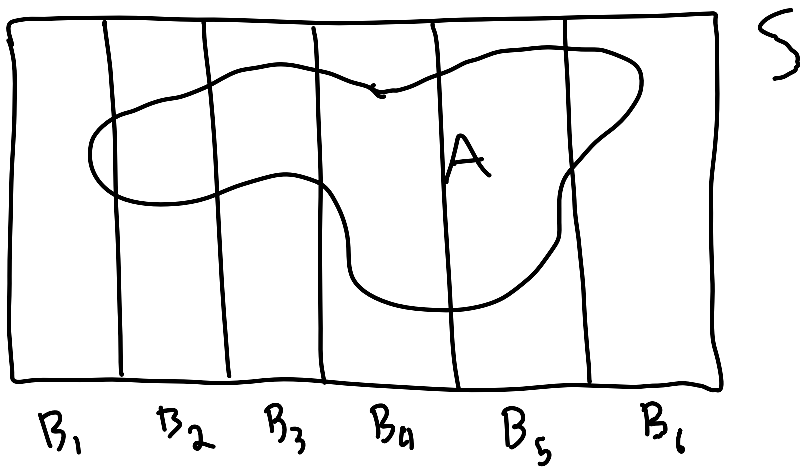
By (**) A^c and B are independent

$\Longrightarrow A^c$ and B^c are independent

Theorem 1.46 (Law of total probability)

Suppose that (S, P) is a probability space and the events $B_1, B_2, \dots, B_n$ partition S (The $B_1, \dots, B_n$ are pairwise disjoint and their union is S). Then for any $A \subseteq S$ we have:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$



Proof:

For any $i \in \{1, 2, \dots, n\}$,

$$P(A|B_i)P(B_i) = \frac{P(A \cap B_i)}{P(B_i)} P(B_i)$$

(definition of conditional probability)

$$= P(A \cap B_i)$$

$$\sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$$

$$= \sum_{i=1}^{n} P(A \cap B_i)$$

($A \cap B_i$ and $A \cap B_j$ are disjoint, countable additivity)

$$= P\left(\bigcup_{i=1}^{n} (A \cap B_i) \right)$$

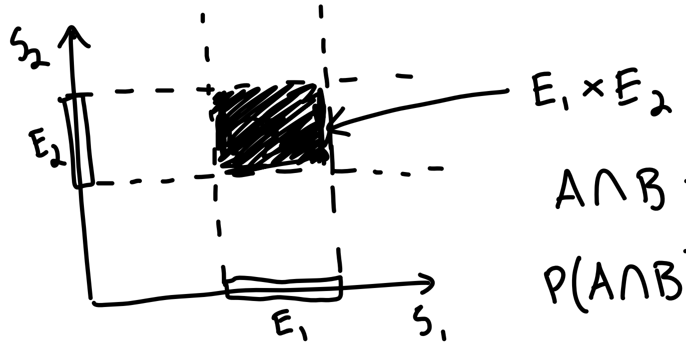
$$= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i \right) \right)$$

$$= P(A \cap S) = P(A) \qquad \square$$

side note: countable additivity proves inclusion-exclusion, but is also a special case of inclusion-exclusion (where the intersections are all empty) :)

Visualizing independent events:

Let $A = E_1$, $B = E_2$ where $E_1 \subseteq S_1$, $E_2 \subseteq S_2$



$$A \cap B = E_1 \times E_2$$

$$P(A \cap B) = P(E_1) P(E_2)$$
$$= P(A) P(B)$$