9/12/25

(in book)

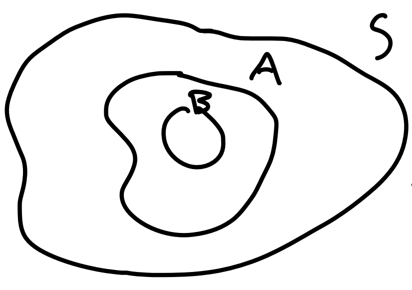
<u>Proposition 1.35:</u> Suppose that S is a sample space, P is a probability function, and we have an event $B \subseteq S$ such that $P(B) \neq 0$. Then the correspondence $A \mapsto P(A|B)$ defines a probability function on S

[Define $P_B(A) = P(A|B)$, where $A \subseteq S$]

1. $0 \leq P(A|B) \leq 1$, $A \subseteq S$

   $0 \leq P_B(A) \leq 1$

2. If $B \subseteq A \subseteq S$, then $P(A|B) = 1$



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

~ given B, we are already inside A

3. If $A_1, A_2, A_3, \ldots$ pairwise disjoint, and put $A = \bigcup_{n=1}^{\infty} A_n$, then

$$P(A|B) = \sum_{n=1}^{\infty} P(A_n|B)$$

Independent Events

$$P(A|B) = P(A)$$ ~ Event B has no influence on Event A

$$P(A \cap B) = P(A) \cdot P(B)$$

Example 1.39: Flip a fair coin twice. $|S| = 4$

$A = \{HT, HH\}$ $B = \{TH, HH\}$ $C = \{TH, HT\}$

which events are independent?

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$ $P(C) = \frac{1}{2}$

$$P(A \cap B) = P(\{HH\}) = \frac{1}{4}$$

$$P(A \cap C) = P(\{HT\}) = \frac{1}{4}$$

$$P(B \cap C) = P(\{TH\}) = \frac{1}{4}$$

all of them are independent!

$$P(A \cap B \cap C) = P(\emptyset) = 0$$

can be pairwise independent without being independent of all

what's going on here?

$S_1 = \{H, T\}$ $\qquad$ $S = S_1 \times S_2$

$S_2 = \{H, T\}$ $\qquad$ $\hookrightarrow$ HT HH

TH TT

$A = \{H\} \times S_2$ $\qquad$ $A \cap B = \{H\} \times \{H\}$

$B = S_1 \times \{H\}$

$$P(A \cap B) = P(\{H\} \times \{H\})$$
$$= P(\{H\}) P(\{H\})$$

Definition 1.40: The sequence of events $A_1, A_2, \ldots, A_n, \ldots$ is called independent if for any $k \geq 1$, and for any $i_1 < \cdots < i_k$, we have:

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k})$$

Example 1.41: Roll a die until the first 6 appears. What is the probability that this happens on the n-th roll?

$A_k = \{\text{roll } k \text{ gives a } 6\}$

$P(A_k) = \frac{1}{6}$ $\qquad$ $P(A_k^c) = \frac{5}{6}$

* Assumption: each roll is independent

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$$P(A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \cap A_n)$$
$$= P(A_1^c)\,P(A_2^c) \cdots P(A_{n-1}^c)\,P(A_n)$$
$$= \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$$

<u>Proposition 1.38</u>: If the events, A, B, are independent, then so are the events A and B^c

<u>Proof</u>: see start of next class (Sep 15)