

9/10/25

Inclusion - Exclusion Principle: Suppose that S is a sample

space with probability function P and we take events

$$A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq S \quad (A_1, A_2, \dots, A_n \text{ need not be disjoint})$$

Then,
$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

remove extra pairwise intersections
$$- \sum_{i < j} P(A_i \cap A_j)$$

add back all triple intersections
$$+ \sum_{i < j < k} P(A_i \cap A_j \cap A_k)$$

remove extra ... intersections
$$- \dots$$

add back intersection...
$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n)$$

Conditional Probability

"If A happens, then what is the probability of B happening?"

Example:

- Roll a pair of dice (6 numbers, fair)
- Let A be the event that the same number is rolled on both dice.
- Let B be the event that the sum of the numbers is 10.
- What is the probability of A given that B has happened?

B happens $\rightarrow$ must be a case in $\{(4,6), (5,5), (6,4)\}$

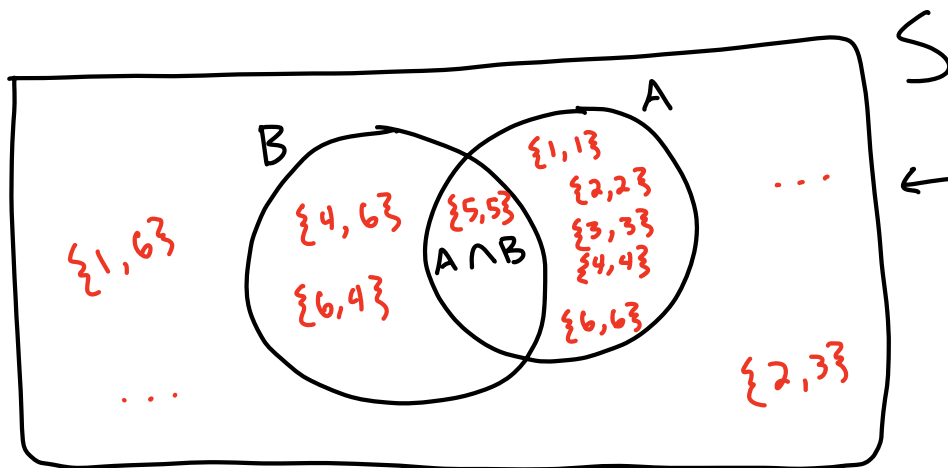
Among these, only $(5,5)$ falls under $A \rightarrow$ Answer: $\frac{1}{3}$

Conditional Probability Definition:

Let A and B be two events. Then the probability of A given B is $P(A|B) = \frac{P(A \cap B)}{P(B)}$ where $P(B) \neq 0$

(continuing dice example)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$



← B has happened means we are inside B
 $\Rightarrow$ we need to measure $A \cap B$ inside B
 $\Rightarrow$ divide the two

some algebra: $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Bob and Alice are playing a gambling game. Each rolls 1 die and the person with the higher number wins. If Alice won, what is the probability that she rolled a 5?

→ a solution:

Let A be the event that Alice wins

Let T be the event of a tie

$$P(T) = \frac{6}{36} = \frac{1}{6}$$

$$P(T^c) = 2P(A)$$

Bob wins = $P(A)$
 $(P(A) + P(B) = 2P(A))$

$$1 - \frac{1}{6} = 2P(A)$$

$$2P(A) = \frac{5}{6} \Rightarrow P(A) = \frac{5}{12}$$

Bob Rolls

	1	2	3	4	5	6
1	tie					
2	/	tie				
3	/	/	tie			
4	/	/	/	tie		
5	X	X	X	X	tie \	/
6	/	/	/	/	/	tie

Alice Wins

Alice Rolls a 5

Let E be the event that Alice Rolls a 5.

$$P(E|A) = \frac{P(E \cap A)}{P(A)} = \frac{\frac{4}{36}}{\frac{15}{36}} = \frac{4}{15}$$