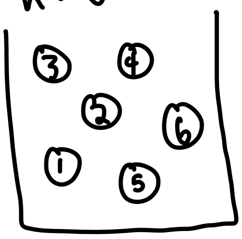
9/3/25

	replacement	no replacement
order	n^k	$(n)_k$
no order	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$

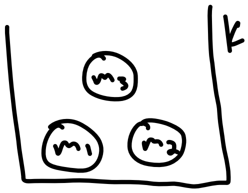
(# of possibilities of n things drawn k times)

$n=6$



Box 1

$k=3$



Box 2

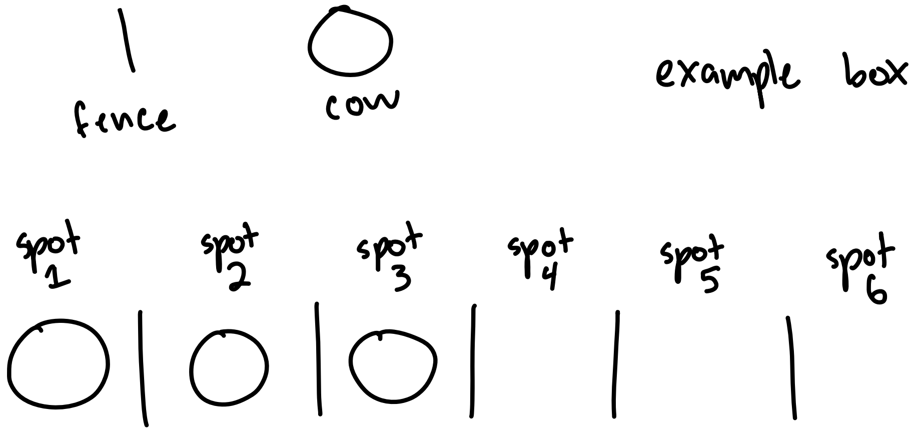
1. Take a ball out from box 1. Put another ball with the same number in box 2. Put the original ball back in box 1, and record how many times ball m_1 has been seen.
2. (repeat for a total of k times)

A the end of this we have:

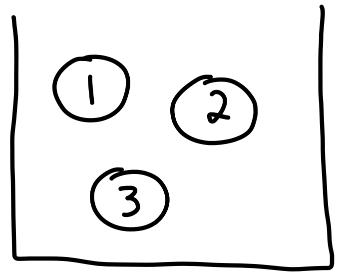
- a box with k numbers in them
- a list $f_1, f_2, \ldots, f_n$ which tells us how many times we have seen a ball with a certain number
  - ↳ f_i is the number of times ball i appears in box 2.

To count how many boxes and lists we have, we need to figure out the number of ways to satisfy:

$$f_1 + f_2 + f_3 + \ldots + f_n = k$$



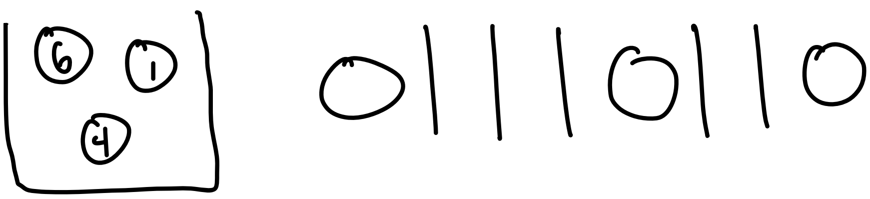
example box 2:



$f_1 = 1$ $\quad f_4 = 0$
$f_2 = 1$ $\quad f_5 = 0$
$f_3 = 1$ $\quad f_6 = 0$

another example:

another example:



Observation: sequence of ○ and | .

- How many ○ appear in each sequence? K
- How many | appear in each sequence? $n-1$
- Length of sequence: $k+n-1$

If I pick the spots for ○ then I am done:

$$\binom{k+n-1}{k}$$

If I pick the spots for | then I am done:

$$\binom{k+n-1}{n-1}$$

or we can do: Take all permutations, $(k+n-1)!$

orderings of 0 : $k!$
(each for which we also have d)
orderings of 1 : $(n-1)!$

Final count is:
$$\frac{(k+n-1)!}{k!(n-1)!}$$

$$\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{k} = \binom{k+n-1}{n-1} = \frac{(k+n-1)!}{k!(n-1)!}$$

In a group of k people, what is the probability that two share a birthday? (assume a year has 365 days)

- all possible birthday options: 365^k
- all possible birthday options with no repetitions:

$$(365)_k = (365)(364)(\cdots)(365-k+1)$$

probability that no two people have the same birthday:
$$\frac{(365)_k}{365^k}$$

Answer: $1 - \frac{(365)_k}{365^k}$

After class: why can't we do this unordered?

i.e. why does $1 - \frac{\binom{n}{k}}{\binom{k+n-1}{k}}$ not work?

☆ Key insight: Unordered multisets are not equally likely.

- we model using ordered sequences because birthdays are independent:
  - person 1: one of n possible outcomes
  - person 2: one of n possible outcomes
  - ...
  - person k: one of n possible outcomes

  n^k total outcomes

  → all ordered outcomes are equal (e.g. AA, AB, BA, AC, CA, BB, ...)

but not all unordered multisets are equal (e.g. {A,A} occurs once vs. {A,B} occurs twice)