8/29/25

(continuity from below)

Recall: If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ are events of some sample space with probability function P, then

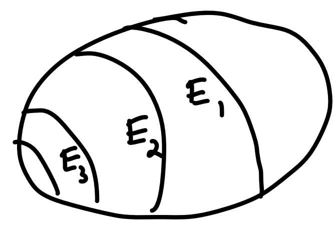
$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$

Notation: $\bigcup_{n=1}^{\infty} E_n = \lim_{n\to\infty} E_n$ $\qquad P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$

<u>continuity from above:</u>

If $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ are events in a probability space S with a probability function P, then

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$



<u>Coin tossing:</u>

Toss a fair coin n times. S_n = All sequences made from H and T of length n

$$S_2 = \{HH, HT, TH, TT\}$$

what is the probability of never getting T?

$\frac{1}{4}$ if $n=2$, $\frac{1}{2^n}$ in general

If I keep flipping a coin until I get T, will I ever stop?

Should be yes.

S = all infinite sequences of H and T

ex - things in S: HTTHHTHTTT ...

HHTTTTHTH ...

only case w/ no tails → HHHHHHHHH ...

Let $E = \{HHHH \cdots H\}$. $P(E) = ?$

- first: how do we define E?

E_1 is the event that the first toss is H

E_2 is the event that the first 2 tosses are H

E_3 " 3 "

$$P(E) = P\left(\bigcap_{n=1}^{\infty} E_n\right) \quad \leftarrow \text{continuity from above}$$

$$= \lim_{n \to \infty} P(E_n)$$

$$= \lim_{n \to \infty} \frac{1}{2^n} = 0$$

<u>Unique product of probability spaces:</u>

Suppose that $S_1, S_2, S_3, \ldots$ are probability spaces with probability functions $P_1, P_2, P_3, \ldots$.

Let $S = S_1 \times S_2 \times S_3 \times \cdots \times S_n \times \cdots$. If for all events $E_1 \subseteq S_1$, $E_2 \subseteq S_2$, $E_3 \subseteq S_3, \ldots, E_n \subseteq S_n$ and a probability function P on S such that

$$P(E_1 \times E_2 \times \cdots \times E_n \times S_{n+1} \times S_{n+2} \times S_{n+3} \times \cdots)$$
$$= P_1(E_1) \cdot P_2(E_2) \cdot P_3(E_3) \cdots P(E_n)$$

then P is unique.

✪ Works in finite case too.

Multiplication Principle:

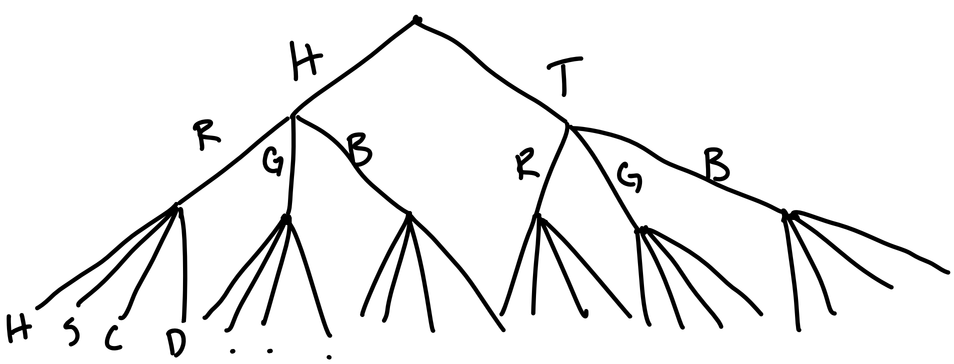
If we perform, in order, r experiments so that the number of possible outcomes on the k-th experiment is n_k, then the number of possible outcomes of this ordered sequence is $n_1 n_2 n_3 \cdots n_r$

$r = 3$ * First toss a coin, H, T

$n_1 = 2$ * Pick a marble out of bag of three, R, G, B

$n_2 = 3$

$n_3 = 4$ * Pick a card from a standard deck, H, C, S, D



total # of outcomes: $n_1 \times n_2 \times n_3 = 2 \times 3 \times 4 = 24$