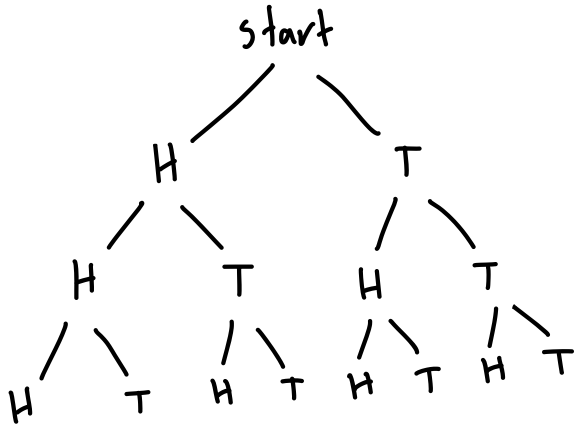
8/27/25

Flip a coin n times. Coin has sides H, T.



$A_1 A_2 A_3 \cdots A_n$

A_i is either H or T

$$\underbrace{\{H, T\} \times \cdots \times \{H, T\}}_{n \text{ times}}$$

$$S_n = \{A_1 A_2 \cdots A_n \text{ such that } A_i \text{ is } H \text{ or } T\}$$

$$S_1 = \{H, T\}$$

$$S_2 = \{HT, HH, TH, TT\}$$

$$|S_3| = 8, \quad |S_4| = 16, \quad |S_n| = 2^n$$

for each choice you have two more choices, for n total flips

What is the probability that we never get tails?

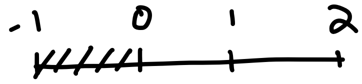
$\hookrightarrow \frac{1}{2^n}$ only 1 sequence all HHH⋯H

(what <u>should</u> happen)

If I flip a coin infinitely many times, then what is the probability of never getting tails?

Intuition says <u>zero</u>

$S = [-1, 2]$



what is the probability of getting a negative number?

$\frac{1}{3}$

what if E = all rational numbers in S?

$\hookrightarrow$ $P(E) = 0$ (intuition based on countably more infinite irrational)

Uniform Probability

- If S is a finite set and $a \in S$ and put $E = \{a\}$

$$P_{unif}(E) = \frac{1}{|S|}$$ uniform distribution

$$P_{unif}(\{a_1, \dots, a_n\}) = \frac{n}{|S|}$$

How do we model things that aren't fair?

S finite

weight function $w : S \to [0, \infty)$

$S = \{B, R\}$

$w(B) = 30$

$w(R) = 60$

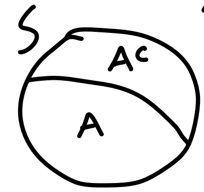
$$P(\{a_1, \dots, a_n\}) = \frac{w(a_1) + \dots + w(a_n)}{\sum_{b \in S} w(b)}$$

$\uparrow$ elements in S, outcomes we want to talk about

$$P(\{B\}) = \frac{30}{30+60}$$

Proposition: Let S be a sample space and P a probability function. Then:

I) $P(A^c) = 1 - P(A)$



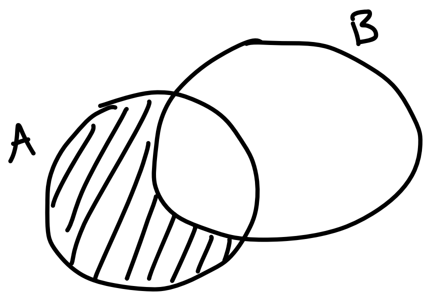
"A complement"
A^c is all elements in S that are not in A

Proof: $S = A \cup A^c$

$P(S) = P(A \cup A^c)$ – finite additivity (they are pairwise disjoint)

$1 = P(A) + P(A^c)$

II) $P(\underset{\text{"without"}}{A \setminus B}) = P(A) - P(\underset{\text{"intersects"}}{A \cap B})$



Proof: $A = (A \setminus B) \cup (A \cap B)$

$P(A) = P(A \setminus B) + P(A \cap B)$

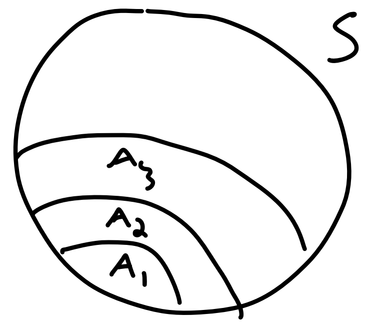
III) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

IV) $P(A^c \cap B^c) = 1 - P(A \cup B)$

V) For all $A, B \subseteq S$ if $A \subseteq B$ then

$P(A) \leq P(B)$

A sequence of events $(A_n)_{n \in \mathbb{N}}$ is called increasing if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$



Notation:

$$\lim_{n \to \infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots$$

Continuity from below:

A sequence of events $(A_n)_{n \in \mathbb{N}}$ that is increasing, then

$$P\left(\lim_{n \to \infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

intuition: (after class) because sequence of events is increasing, A_n will always contain all A_i for $i < n$, so the probability of the union of all A_i for $i = 0, 1, \ldots, n$ is the same as the probability of A_n (and as n approaches infinity)