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Probability?

rolling dice, rain

- likelyhood of an <u>event</u>; probability is between 0 and 1

- chance getting an even number:

  options: 1, <u>2</u>, 3, <u>4</u>, 5, <u>6</u> $\quad \frac{3}{6} = \frac{1}{2}$] * Application of probability

- Probability is a modern field in pure math with applications to combinatorics, analysis, number theory, etc.
- Applications in economics, computer science, etc.

* Sample Space, in math this is just a set S.

  sample: $S = \{1, 2, 3, 4, 5, 6\}$

* Event, set $E \subseteq S$

$$E = \{2, 4, 6\}$$

- We need some way to <u>measure</u> events inside some sample space
- A probability function measures how much an event takes up from the sample space. e.g. $\frac{3}{6} = 0.5$ (intuitive)

Probability function formally:

consider a sample space S.

We say that P is a probability function if it takes events as inputs and outputs real numbers. P must also satisfy the following: $E \subseteq S$

I (Positivity) $0 \leq P(E) \leq 1$

...

II (Normalization) $P(\emptyset) = 0$, $P(S) = 1$

III (countable Additivity) If $E_1, E_2, \dots$ are pairwise disjoint events then

$$P\left(\bigcup_{n\geq 1} E_n\right) = \sum_{n \geq 1} P(E_n)$$

$$= \sum_{n=1}^{\infty} P(E_n)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} P(E_n)$$

ex: $S = \{1,2,3,4,5,6\}$

$E_1 = \{1\}$, $E_2 = \{2\}$, $E_3 = \{3\}$

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

- countable additivity implies finite additivity

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) \Rightarrow P\left(\bigcup_{n=1}^{k} D_n\right) = \sum_{n=1}^{k} P(D_n)$$

proof: Some events $D_1, \dots, D_k \subseteq S$

define $D_{k+1} = \emptyset$, $D_{k+2} = \emptyset$, $D_{k+3} = \emptyset$, ...

$$\bigcup_{n=1}^{k} D_n = \bigcup_{n=1}^{\infty} D_n$$

$$P\left(\bigcup_{n=1}^{k} D_n\right) = P\left(\bigcup_{n=1}^{\infty} D_n\right) = \sum_{n=1}^{\infty} P(D_n) = \lim_{i\to\infty} \sum_{n=1}^{i} P(D_n)$$

$$= \sum_{n=1}^{k} P(D_n) \quad \square$$

- In general, these definitions are not as good as possible.
  - How do you assign a probability to an event? Events are subsets
  - Can you assign a probability to every $E \subseteq S$? No
    - ↳ Measure spaces, and answering the question of what is measurable
    - ↳ probability spaces are the nicest measure spaces