4/9/25

Quiz: 1) A is similar to C if $A = BCB^{-1}$ for some invertible B

2) $T(b_1) = 3b_1$, $T(b_2) = -5b_2$

a. $\lambda = 3, -5$

b. $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$ matrix for T w.r.t. B-coords

c. $M_{B\to S}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ b_1

$M_{B\to S}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$ b_2

$M_{B\to S} = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}$

d. $A\vec{v}$: ① take $\vec{v}$ to B-coords, ② apply T in B-coords, ③ go back to std. coords

$$A = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}3 & 0\\0 & -5\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}^{-1}$$

(introduced last class:)

Thm

A $m \times m$. If $\lambda_m, \dots, \lambda_m$ are distinct e-vals corresponding to $\vec{v}_1, \dots, \vec{v}_m$, then $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

Pf. Suppose $\vec{v}_1, \dots, \vec{v}_m$ are dependent.

Then for some k, $\vec{v}_k = c_1\vec{v}_1 + \dots + c_{k-1}\vec{v}_{k-1}$ — choose smallest such k

since $v_k \neq 0$, some $c_i \neq 0$

Multiply by A to get: $\lambda_k \vec{v}_k = c_1\lambda_1\vec{v}_1 + \cdots + c_{k-1}\lambda_{k-1}\vec{v}_{k-1}$

Multiply by λ_k to get: $\lambda_k \vec{v}_k = c_1\lambda_k\vec{v}_1 + \ldots + c_{k-1}\lambda_k\vec{v}_{k-1}$

Subtract the two: $0 = (\lambda_1 - \lambda_k)c_1\vec{v}_1 + \ldots + (\lambda_{k-1} - \lambda_k)c_{k-1}\vec{v}_{k-1}$

$$\Longrightarrow (\lambda_i - \lambda_k)c_i = 0 \quad \text{for all } i$$

$$\Longrightarrow \text{some } \lambda_i = \lambda_k \quad \rightarrow\leftarrow$$

$\square$

Cor. If $n \times n$ A has n distinct e-vals, then A is diagonizable.

Cor. Vectors from distinct eigenspaces are linearly independent

Ex: Diagonalize and compute A^k.

$$A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$$

Soln.

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10 = 0$$

$$= \lambda^2 - \lambda - 12 + 10 = 0$$

$$= (\lambda - 2)(\lambda + 1) = 0$$

$\lambda = 2, \ \lambda = -1$

$\text{Nul}(A - 2I) = \text{Nul}\left(\begin{bmatrix} -5 & 5 \\ -2 & -2 \end{bmatrix}\right)$

$\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = x_2$, $x_2 = x_2$ $\quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\text{Nul}(A + I) = \text{Nul}\left(\begin{bmatrix} -2 & 5 \\ -2 & 5 \end{bmatrix}\right)$

$\longrightarrow \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix}$ $\quad 2x_1 = 5x_2$, $x_2 = x_2$ $\quad v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

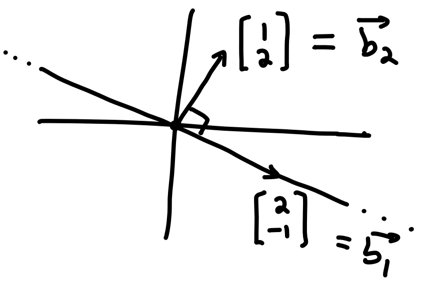
$$A = \underbrace{\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}}_{\mathcal{B} \to \mathcal{S}} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{A \text{ in} \\ \mathcal{B}\text{-coords}}} \underbrace{\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}^{-1}}_{\mathcal{S} \to \mathcal{B}} = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2/3 & 5/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} -2/3 & 5/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2^{k+1} + 5(-1)^k & 5 \cdot 2^k + 5(-1)^{k+1} \\ -2^{k+1} + 2(-1)^k & 5 \cdot 2^k + 5(-1)^{k+1} \end{bmatrix}$$

cool!

Ex: Find the matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting over $x + 2y = 0$

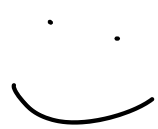


$T(b_1) = b_1$ ~ on the line!

$T(b_2) = -b_2$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ e-vec w/ e-val 1

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ e-vec w/ e-val -1



$$A = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}_{\mathcal{B} \to \mathcal{S}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{in } \mathcal{B}\text{-coords}} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}^{-1}}_{\mathcal{S} \to \mathcal{B}}$$

$$A = \ldots (\text{inverse \& multiply}) \ldots = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$