4/7/25 (contd. from prev class)

3. (transitive) If A is similar to C, and C is similar to E, then A is similar to E.

Pf: $A = BCB^{-1}$ $\quad C = DED^{-1}$

$$A = BDED^{-1}B^{-1} = (BD)E(BD)^{-1}$$

Rmk: This shows that similarity is an equivalence relation

Goal: Write matrices as similar to diagonal matrices.

Ex: $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}_{=\vec{b}_1}, \underbrace{\begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{=\vec{b}_2} \right\}$

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(\vec{b}_1) = 3\vec{b}_1 \quad T(\vec{b}_2) = -2\vec{b}_2$

$[\vec{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[\vec{b}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then in $\mathcal{B}$-coordinates, T has matrix
$$\begin{array}{cc} [T(b_1)]_{\mathcal{B}} & [T(b_2)]_{\mathcal{B}} \\ \end{array}$$
$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

So, std matrix of T in std. coordinates are:

$$\underbrace{\begin{bmatrix} -2 & 3 \\ 3 & -1 \end{bmatrix}}_{\mathcal{B}\text{-coords to standard coords}} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}_{T \text{ in } \mathcal{B}\text{-coords}} \underbrace{\begin{bmatrix} -2 & 3 \\ 3 & -1 \end{bmatrix}^{-1}}_{\text{std coords to } \mathcal{B}\text{-coords}}$$

Ex: Are $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ similar?

No, ① different e-vals

② $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \stackrel{?}{=} B \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B^{-1} = BB^{-1} = I$ $\times$

so the identity matrix is only similar to itself

Key example:

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is a diagonal matrix

- The e-vals are $\lambda_1, \dots, \lambda_n$ and the e-vecs are $e_1, \dots, e_n$

Thm (on HW)

Similar matrices have the same e-vals and dimensions of eigenspaces.

Pf: Similar matrices represent same linear transformation w.r.t. different bases. e-vals, dims of e-spaces are properties of linear transformation independent of basis

<u>Defn</u>: A square matrix is <u>diagonalizable</u> if it is similar to a diagonal matrix

<u>Cor</u>: A is <u>diagonizable</u> $\iff$ there is a basis for $\mathbb{R}^n$ of e-vecs of A

<u>Ex</u>: Show that $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

e-vecs: $\vec{e}_1, \vec{e}_2$ e-vecs: $\vec{e}_1, \vec{e}_2$

<u>Soln</u>: want to change order of basis

$$C\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = CBC^{-1} = CBC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

<u>Ex</u>: Are $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ similar?

No: dim($\lambda = 2$ e-space of A) = 2

dim($\lambda = 2$ e-space of B) = 1 B is not diagonizable

$\hookrightarrow$ <u>e-vecs</u> $B - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $x_2 = 0$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Rmk: Suppose $A = CDC^{-1}$

$$A^k = CDC^{-1}CDC^{-1} \cdots CDC^{-1}$$
$$= CD^kC^{-1}$$

Thm (independence of e-vecs)

A $n \times n$. If $\lambda_1, \ldots, \lambda_n$ are distinct e-vals corr. to e-vecs $\vec{v}_1, \ldots, \vec{v}_n$, then $\vec{v}_1, \ldots, \vec{v}_n$ are lin. ind.