4/4/25

<u>Def^n</u>: The matrix for the linear transformation

$$T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$$ is called the <u>change of basis</u> matrix from the standard basis to the basis $\mathcal{B}$.

Ex: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

bad cursive "s" $\rightarrow$ $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$M_{\mathcal{B}\to\mathcal{S}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad M_{\mathcal{B}\to\mathcal{S}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$M_{\mathcal{B}\to\mathcal{S}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \hookrightarrow M_{\mathcal{S}\to\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

<u>Prop</u>: The change of basis matrix is invertible.

<u>Pf</u> $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \Rightarrow M_{\mathcal{B}\to\mathcal{S}} = [\vec{b}_1 \ \cdots \ \vec{b}_n]$

linearly independent & span $\mathbb{R}^n$ (by def)

$\vec{b}_i$ are independent $\Rightarrow M_{\mathcal{B}\to\mathcal{S}}$ is invertible

$\square$

Q: Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. What is the inverse of T w/

$$T(\vec{v}) = [\vec{v}]_{\mathcal{B}}$$

A: $T(\vec{b}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\vec{b}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T^{-1}(\vec{e}_1) = \vec{b}_1$, $T^{-1}(\vec{e}_2) = \vec{b}_2$

Summary $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$

$M_{\mathcal{B}\to\mathcal{S}}$ taking $\mathcal{B}$ coordinates to standard coordinates:

$$M_{\mathcal{B}\to\mathcal{S}} = \begin{bmatrix} \vec{b}_1, \dots, \vec{b}_n \end{bmatrix} \quad \text{since } M_{\mathcal{B}\to\mathcal{S}}\vec{e}_i = \vec{b}_i$$

$M_{\mathcal{S}\to\mathcal{B}}$ taking standard coordinates to $\mathcal{B}$ coordinates:

$$M_{\mathcal{S}\to\mathcal{B}} = M_{\mathcal{B}\to\mathcal{S}}^{-1}$$

Q: why is this useful?

A: Matrices can be simpler w/ different basis!

Ex: A matrix w/ e-vals -1 (e-vec $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$),

2 (e-vec $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$)

Let's write $T(\vec{v}) = A\vec{v}$ w.r.t. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

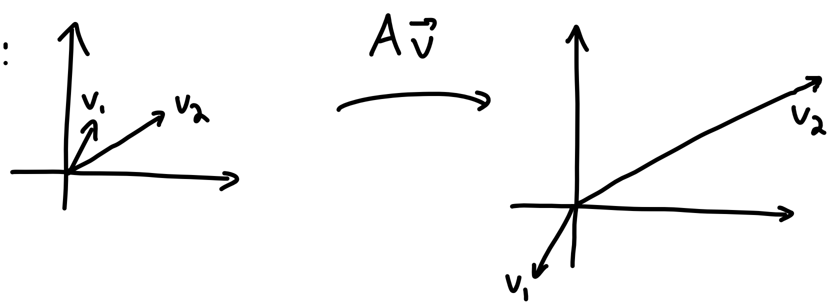
We <u>want</u> D w.r.t this basis.

$$De_1 = \left[A\vec{b}_1 \right]_{\mathcal{B}} = \left[-\vec{b}_1 \right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$De_2 = \left[A\vec{b}_2 \right]_{\mathcal{B}} = \left[2\vec{b}_1 \right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

T:



To find A: use $A = M_{\mathcal{B}\to\mathcal{S}} D M_{\mathcal{S}\to\mathcal{B}}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 4 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix}$$

Let's check:

$$A\vec{v}_1 = \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -\vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 2\vec{v}_2$$

:)

Defn: Two $n \times n$ matrices are <u>similar</u> if there is an invertible B w/ $A = BCB^{-1}$

Ex: $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix}$ because

$$\underbrace{\begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}}_{B^{-1}}$$

Rmk: Similar matrices represent the same linear transformations w.r.t. a different basis

Basic Properties

1. (reflexive) A is similar to A

   Pf: $A = IAI^{-1}$

2. (symmetric) If A is similar to C, then C is similar to A

   Pf: $A = BCB^{-1}$

   $B^{-1}AB = C$

   $C = B^{-1}A(B^{-1})^{-1}$

(contd. next time!)