4/30/25

- Quiz : 1) orthogonal matrix is matrix whose columns are orthonormal

2) $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ orthonormal basis

$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ $\qquad W = \text{span}\{b_1, b_2\}$

$v_W = 2b_1 + 3b_2$

$2 = v \cdot b_1 = c_1$

$3 = v \cdot b_2 = c_2$ $\qquad [\vec{v}]_B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ $\qquad [v_W]_B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$4 = v \cdot b_3 = c_3$

<u>Ex:</u>

Ex: $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, find a formula for A^k

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) + 2 = 0$$

$$18 - 9\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 9\lambda + 20 = 0$$

$$(\lambda - 4)(\lambda - 5) = 0$$

$$\lambda = 4, 5$$

$\text{Nul}(A - 4I) = \text{Nul}\left(\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\right)$

$\hookrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = \frac{1}{2}x_2$, $x_2 = x_2$ $\qquad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\text{Nul}(A - 5I) = \text{Nul}\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\right)$

$\hookrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2$, $x_2 = x_2$ $\qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{\mathcal{B} \to \mathcal{S}} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}}_{[T]_{\mathcal{B}}} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}}_{\mathcal{S} \to \mathcal{B}}$$

$$A^k = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

<u>Ex:</u> $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \underline{1} & 2 & 0 & 13 & 33 \\ 0 & 0 & \underline{1} & -6 & -15 \end{bmatrix}$

pivot columns

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 13 \end{bmatrix} \right\} = \mathbb{R}^2 \quad , \quad \text{rk} A = 2$$

Nul(A) :

$$x_1 = -2x_2 - 13x_4 - 33x_5$$
$$x_2 = x_2 \qquad x_4 = x_4$$
$$x_3 = 6x_4 + 15x_5 \qquad x_5 = x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(contd.)

Let $T(\vec{v}) = A\vec{v}$ $\quad$ T injective? No!

T inj. $\iff$ $\dim \mathrm{Nul}(A) = 0$

T surjective? Yes!

T surj. $\iff$ $\mathrm{rk}\, A = m$

<u>rank-nullity</u>

$$\underbrace{\text{rank } A}_{\dim \mathrm{Col}(A)} + \dim \mathrm{Nul}(A) = n$$

e.g. (above): $\quad 2 + 3 = 5$

☆ Review: Solutions to $A\vec{x} = \vec{b}$ look like translates of $\mathrm{Nul}\, A$

Suppose A $n \times n$. Then A is invertible if & only if:

- $\mathrm{RREF}(A) = I_n$
- pivot in every col
- product of elementary matrices
- $\det(A) \neq 0$
- every e-val $\neq 0$
- $\mathrm{Col}(A) = \mathbb{R}^n$
- $A\vec{x} = \vec{b}$ always consistent