4/28/25

Thm Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$. Then

$$[\vec{v}]_{\mathcal{B}} = \left(\frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}, \dots, \frac{\vec{v}\cdot\vec{v}_n}{v_n\cdot v_n} \right)$$

Pf. Write $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

$\vec{v}\cdot\vec{v}_i = c_i\vec{v}_i\cdot\vec{v}_i$ (all others go to zero)

$\Rightarrow \vec{c}_i = \dfrac{\vec{v}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$ $\square$

Defn: An orthogonal matrix is an $n\times n$ matrix whose columns form an orthonormal set

HW: for A orthogonal, $A^{-1} = A^T$

Idea: Multiplying by an orthogonal matrix preserves lengths and angles

Prop: If A is orthogonal, then for all $\vec{u}, \vec{v} \in \mathbb{R}^n$,

① $A\vec{v}\cdot A\vec{u} = \vec{v}\cdot\vec{u}$

② $\|A\vec{v}\| = \|\vec{v}\|$

Pf. ① $A\vec{v} \cdot A\vec{u} = (A\vec{v})^T A\vec{u}$

$$= \vec{v}^T A^T A \vec{u} = \vec{v}^T I_n \vec{u}$$

$$= \vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

② $\|A\vec{v}\| = \sqrt{A\vec{v} \cdot A\vec{v}}$ (follows from previous)

$$= \sqrt{\vec{v} \cdot \vec{v}} = \|\vec{v}\|$$

Rmk: Gram-Schmitt algorithm helps you find orthogonal bases

Singular Value Decomposition (SVD)

SVD hope: Given an $m \times n$ matrix A, what if we could "diagonalize" it?

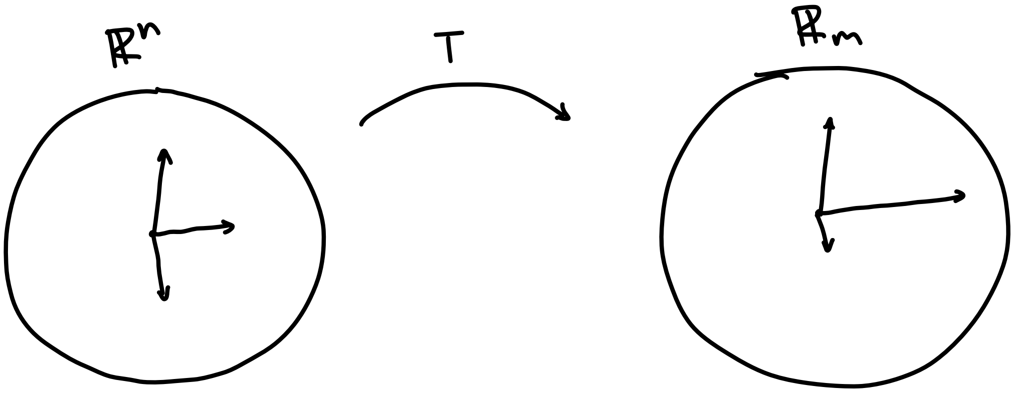
Ex: $\begin{bmatrix} 7 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$ (0's everywhere except for diagonal)

Thm Any $m \times n$ matrix can be written as $A = UDV$

where:
- U is $m \times m$ orthogonal
- V is $n \times n$ orthogonal

- D is $m \times n$ with zeros everywhere except along diagonal, all entries non-negative

$T(\vec{v}) = A\vec{v}$



V changes basis on $\mathbb{R}^n$ U changes basis on $\mathbb{R}^m$

Data Analysis Idea

- Largest entries of D are most important, so can compress data and remove noise by replacing some diagonal entries of D w/ 0.

Ex:

$$D = \begin{bmatrix} 100.2 & & & & \\ & 39.7 & & & \\ & & 5.1 & & \\ & & & 0.1 & \\ & & & & 0.07 \end{bmatrix} \longrightarrow \begin{bmatrix} 100.2 & & & & \\ & 39.7 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

Thm (spectral theorem)

If A is a symmetric matrix, then A has an orthogonal basis of eigenvectors, and all eigenvalues are in $\mathbb{R}$.

Fact:

$A^T A$ for <u>any</u> matrix A has all eigenvalues nonnegative.