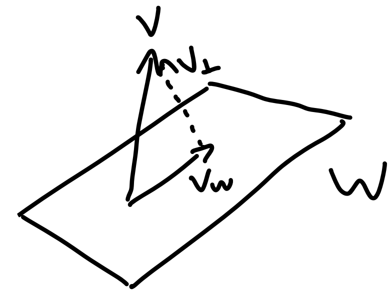
4/23/25

- Quiz (no recap)
  - $A^TA\vec{c} = A^T\vec{v}$ always has soln. for $\vec{c}$!
  - look at proofs from prev. class(es)



Thm

Let $\{v_1, \dots, v_k\}$ be a basis for W, $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be a basis for $W^\perp$. Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

Pf. $\dim W + \dim W^\perp = n$, so ETS $\vec{v}_1, \dots, \vec{v}_n$ lin. ind.

suppose $c_1\vec{v}_1 + \dots c_n\vec{v}_n = \vec{0}$

$$\Rightarrow \underbrace{c_1\vec{v}_1 + \dots + c_k\vec{v}_k}_{\in W} = -(\underbrace{c_{k+1}\vec{v}_{k+1} + \dots c_n\vec{v}_n}_{\in W^\perp})$$

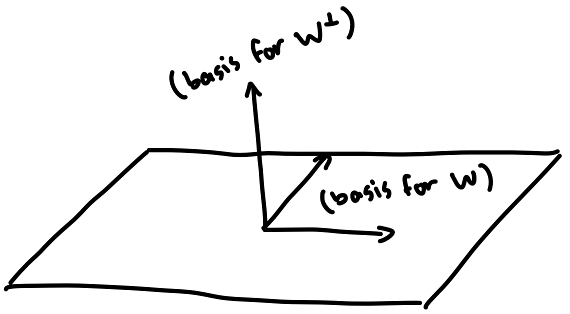
since $W \cap W^\perp = \{\vec{0}\}$, $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$

$$\Rightarrow c_1 = \dots = c_k = 0$$

and $c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n = 0$

$$\Rightarrow c_{k+1} = \dots = c_n = 0$$

$\square$



Cor. Let $\vec{v} \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ subspace.

Then $\vec{v}$ can be written uniquely as

$$\vec{v} = \vec{v}_W + \vec{v}_\perp \quad \text{for } \vec{v}_W \in W, \ \vec{v}_\perp \in W^\perp$$

Pf. Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for W,

$\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be a basis for $W^\perp$.

Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

So, $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

$$\vec{v}_W = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

$$\vec{v}_\perp = c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n$$

$\Longrightarrow$ can write:

$$\vec{v} = \underbrace{\vec{v}_W}_{\in W} + \underbrace{\vec{v}_\perp}_{\in W^\perp}$$

To see uniqueness, suppose

$\vec{v} = \vec{v}_W + \vec{v}_\perp = \vec{u}_W + \vec{u}_\perp$ w/ $\vec{u}_W, \vec{v}_W \in W$, $\vec{u}_\perp, \vec{v}_\perp \in W^\perp$

$$\underbrace{\vec{v}_W - \vec{u}_W}_{\in W} = \underbrace{\vec{u}_\perp - \vec{v}_\perp}_{\in W^\perp}$$

since $W \cap W^\perp = \{0\}$, $\vec{v}_W - \vec{u}_W = 0 = \vec{u}_\perp + \vec{v}_\perp$

$\square$

Ex:

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \qquad A^T\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$A^TA\vec{c} = A^T\vec{v} \longrightarrow \left[\begin{array}{cc|c} 5 & 2 & 3 \\ 2 & 2 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{array}\right]$$

$$\vec{v}_W = A\vec{c}$$

$$A\vec{c} = \begin{bmatrix} 4/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \vec{v}_W \qquad \vec{v}_\perp = \vec{v} - \vec{v}_W = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

check: $\vec{v}_\perp \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 = \vec{v}_\perp \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Ex: (calc III formula proof)

$W = \text{span}\{\vec{u}\}$. Then $A = [\vec{u}]$ and $A^TA = [\vec{u} \cdot \vec{u}]$

$$A^TA\vec{c} = A^T\vec{v}$$

$$[\vec{u} \cdot \vec{u}]\vec{c} = \vec{u} \cdot \vec{v}$$

$$\vec{c} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}$$

$$A\vec{c} = \vec{v}_W = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}$$

Linear Regression

Setting: Have some data that we think satisfies $y = c_1 x_1 + \dots + c_n x_n$ where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the input, and y is the output.

Rmk: If you want $y = c_1 x_1 + \dots + c_n x_n + b$, can introduce "dummy" variable x_{n+1} always equal to 1.

Setting (ctd) Have a bunch of data pts $(\vec{x}^i, y_i)$ and want to figure out the c_i that "best" fit the data.

$$A = \begin{bmatrix} \vec{x}^1 \\ \vdots \\ \vec{x}^m \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Want:
$A\vec{c} = \vec{y}$ or at least
$A\vec{c} \approx \vec{y}$

Soln: Let $W = \mathrm{Col}(A)$, write $\vec{y} = \vec{y}_W + \vec{y}_\perp$.

Then we can solve $A\vec{c} = \vec{y}_W$ using ortho. proj.,

gets us the $\vec{c}$ w/ $\|\vec{y} - A\vec{c}\|$ minimized,

i.e. $\sum (y_i - \vec{x}^i \cdot \vec{c})^2$ minimized.