4/16/25

<u>Quiz</u>: 2) $\vec{v} = \begin{bmatrix} r \\ 25 \\ 2r-15 \end{bmatrix}$ $W = \text{span}\left\{ \begin{bmatrix} r \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} r \\ 1 \\ 2 \end{bmatrix} \right\}$ when is $\vec{v} \in W^\perp$?

- could do $W^\perp = \text{Nul}(A^T)$, $W = \text{Col}(A)$
$A = [w_1\ w_2]$

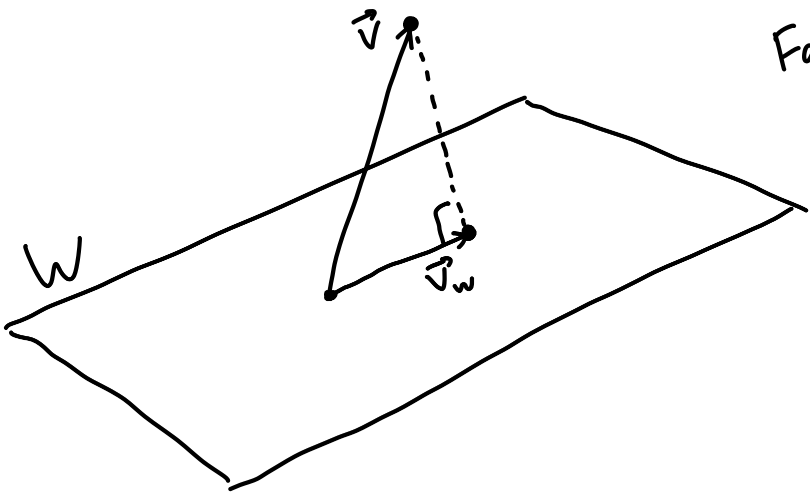
- easier to check $\vec{v} \cdot \vec{w}_1 = 0 = v \cdot \vec{w}_2$

$\vec{v} \cdot \vec{w}_1 = 0 \downarrow r = \pm 5$, $v \cdot \vec{w}_2 = 0 \downarrow r = -5, 1$ $\Rightarrow r = -5$

# <u>Projection</u>

<u>Idea</u>: have vector $\vec{v}$, subspace $W$

want to know: what point of $W$ is closest to $\vec{v}$?




Fact: closest point in $W$ to $\vec{v}$ is $\vec{v}_W$, where $(v - v_W) \in W^\perp$

<u>Goal</u>: Systematic way to find $\vec{v}_W$

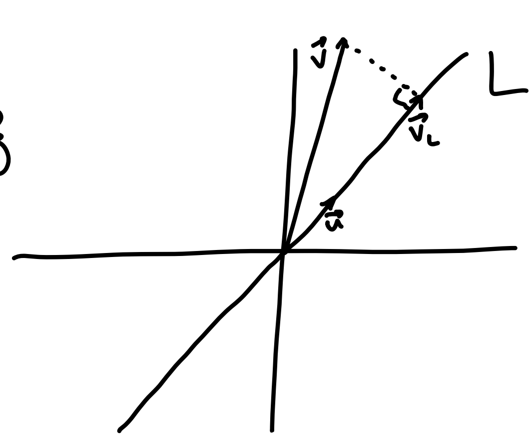
<u>Fact</u>: (HW) $W \cap W^\perp = \{\vec{0}\}$

<u>Cor</u>: Let $\vec{v} \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ subspace. Then $\vec{v}$ can be written uniquely as $\vec{v} = \vec{v}_W + \vec{v}_\perp$ where $\vec{v} \in W$, $\vec{v}_\perp \in W^\perp$

**Defn:** $W \subseteq \mathbb{R}^n$ subspace. The decomposition $\vec{v} = \vec{v}_W + \vec{v}_\perp$ is called the <u>orthogonal decomposition</u> of $\vec{v}$ w.r.t. $W$

**Defn:** $\vec{v}_W$ is called the <u>orthogonal projection</u> of $\vec{v}$ onto $W$.

(from Calc III:)




Projecting $\vec{v}$ onto $L = \text{span}\{\vec{u}\}$

$$\hookrightarrow \quad \vec{v}_L = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

Q: How to do this when $L$ is not a line?

**Thm**

Let $\vec{b} \in \mathbb{R}^n$, $\quad W = \text{Col}(A)$

Then $\vec{b}_W = A\vec{c}$, where $\vec{c}$ is any solution to the consistent system $A^T A\vec{c} = A^T\vec{b}$

**Recap:** To project $\vec{b}$ onto $W$:

1) find $A$ w/ $\text{Col}(A) = W$
2) solve $A^T A\vec{c} = A^T\vec{b}$ for $\vec{c}$
3) answer is $\vec{b}_W = A\vec{c}$

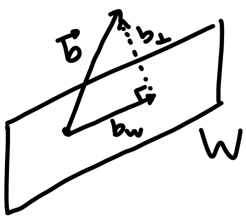
<u>Ex:</u> $W = \text{span}\left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$, $\quad b = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \vec{b}_W = ?$

$$A = \begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix} \qquad A^TA = \begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix} \qquad A^TA\vec{c} = A^T\vec{b} \longrightarrow \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\vec{c} = \begin{bmatrix}-2\\3\end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cc|c}2 & 1 & -2\\1 & 2 & 3\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c}1 & 0 & -\frac{7}{3}\\0 & 1 & 8/3\end{array}\right] \qquad \vec{c} = \begin{bmatrix}-7/3\\8/3\end{bmatrix}$$

$$\vec{b}_W = A\vec{c} = \begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}-7/3\\8/3\end{bmatrix} = \begin{bmatrix}1/3\\8/3\\7/3\end{bmatrix} \qquad \vec{b}_\perp = \vec{b} - \vec{b}_W = \begin{bmatrix}2/3\\-2/3\\2/3\end{bmatrix}$$




<u>check:</u>

$$b_\perp \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix} = 0$$

$$b_\perp \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 0$$

<u>Pf.</u> write $\vec{b} = \vec{b}_W + \vec{b}_\perp$. Since $\vec{b}_W \in W$, there is some $\vec{c}$ with $\vec{b}_W = A\vec{c}$.

$$\vec{b} = \vec{b}_W + \vec{b}_\perp = A\vec{c} + b_\perp, \text{ so}$$

$$A^T\vec{b} = A^TA\vec{c} + \underbrace{A^Tb_\perp}_{=0 \text{ (in Null space)}} = A^TA\vec{c}$$ Thus, system has a solution.

Now suppose $A^TA\vec{c} = A^T\vec{b}$.

Then $A^T(A\vec{c} - \vec{b}) = 0$ so $\vec{b} = \underbrace{A\vec{c}}_{\in W} + \underbrace{(A\vec{c} - \vec{b})}_{\in W^\perp}$

$\Rightarrow A\vec{c} = \vec{b}_W$, $\quad A\vec{c} - \vec{b} = \vec{b}_\perp$

$\square$