4/14/25

Inner Products

Recall: The dot product of $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$u_1 v_1 + \cdots + u_n v_n = u^T v$$

It also has nice algebra properties, like

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}, \quad \vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2, \quad \text{etc.}$$

New properties: $\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2 \geq 0$

$$\vec{v} \cdot \vec{v} = 0 \iff \vec{v} = \vec{0}$$

Defn: The length of $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Rmk: Much of what we do in this class is independent of choice of basis. The dot product is not! If you do data science, make sure to normalize variables in a way that makes sense for your problem

Prop: $\|c\vec{v}\| = |c| \, \|\vec{v}\|$

Pf: $\sqrt{c\vec{v} \cdot c\vec{v}} = \sqrt{c^2 \|v\|^2} = \sqrt{c^2} \|v\| = |c| \, \|v\|$

Ex: $\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\left\| \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

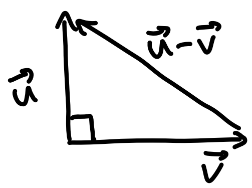
Defn: $\vec{v} \in \mathbb{R}^n$ is a unit vector if $\|\vec{v}\| = 1$.

Ex: unit vector of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Defn: $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal (or perpendicular) if $\vec{u} \cdot \vec{v} = 0$. We write $\vec{u} \perp \vec{v}$

Rmk: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, θ angle between $\vec{u}, \vec{v}$

Prop: $\vec{u} \cdot \vec{v} = 0 \iff \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2$



Pf:
$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} \end{aligned}$$

$\square$

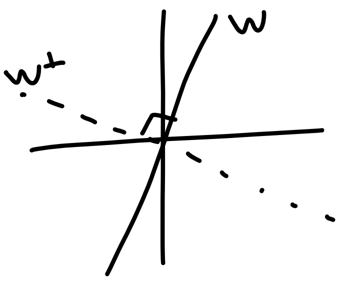
must be zero

Orthogonal Complements

Defn: Let $W \subseteq \mathbb{R}^n$ be a subspace. The <u>orthogonal complement</u> of W, written $W^\perp$, is

$$W^\perp = \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}.$$

Ex: $W = \{ y = 7x \}$ what is $W^\perp$?



$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 7 \end{bmatrix} \right\}$$

$$W^\perp = \left\{ \vec{v} \mid \begin{bmatrix} 1 \\ 7 \end{bmatrix} \cdot \vec{v} = 0 \right\}$$

$$= \{ \vec{v} \mid [1 \ \ 7]\vec{v} = 0$$

$$= \text{Nul}([1 \ \ 7]) \qquad \begin{aligned} x + 7y &= 0 \\ x &= -7y \end{aligned}$$

$$W^\perp = \text{span}\left\{ \begin{bmatrix} -7 \\ 1 \end{bmatrix} \right\}$$

Prop: $W^\perp$ is a subspace

<u>Pf.</u> 1) $\vec{0} \in W^\perp$ since $\vec{0} \cdot \vec{w} = 0$ always

2) let $\vec{u}, \vec{v} \in W^\perp$. Then for any $\vec{w} \in W$,

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0$$

3) Let $\vec{u} \in W^\perp$, $c \in \mathbb{R}$. Then for any $w \in W$,

$$w \cdot (c\vec{u}) = 0 = c\,\vec{w} \cdot \vec{u} = 0$$

$\square$

<u>Prop:</u> Suppose $W = \mathrm{Col}(A)$. Then $W^\perp = \mathrm{Nul}(A^T)$

<u>Pf.</u> Let $A = [\vec{v}_1, \ldots, \vec{v}_k]$.

Then $\vec{v} \in W^\perp \iff \vec{v}_i \cdot \vec{v} = 0$ for all i, since

$$v \cdot \left(\sum c_i \vec{v}_i\right) = \sum c_i \vec{v} \cdot \vec{v}_i$$

Thus, $\vec{v} \in W^\perp \iff \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} \vec{v} = 0 \iff \vec{v} \in \mathrm{Nul}(A^T)$

$\square$

<u>Ex:</u> Compute $W^\perp$ for $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & -2 \\ 7 & 3 \\ 2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 7 & 2 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 7 & 2 \\ 0 & 17 & 5 \end{bmatrix}$$

$$x_1 = -7x_2 - 2x_3 \qquad 17x_2 = -5x_3$$

$$x_1 = 35 - 2 \cdot 17 = 1 \qquad x_2 = -\frac{5}{17}x_3$$

$$\begin{bmatrix} 1 \\ -5 \\ 17 \end{bmatrix}$$

$$W^\perp = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -5 \\ 17 \end{bmatrix} \right\}$$

Prop. If $W \subseteq \mathbb{R}^n$ is a subspace,

$$\dim W + \dim W^{\perp} = n$$

Pf. Let $\vec{v}_1, \dots, \vec{v}_k$ be a basis for W.

Then $W^{\perp} = \mathrm{Nul}\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \mathrm{Nul}(A^T)$.

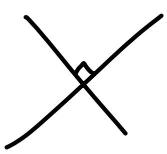
Since $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, as we row reduce we never get row of all zeros

$\Rightarrow k$ pivots in A^T

$\Rightarrow \dim \mathrm{Nul}(A^T) = n - k$

Ex: In $\mathbb{R}^2$: W = line

$W^{\perp}$ = line



In $\mathbb{R}^3$: W = line

$W^{\perp}$ = plane