4/11/25

Ex: Are $\begin{bmatrix} 1 & 7 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ -5 & -4 & -1 \end{bmatrix}$ similar?

Yes: - distinct e-vals $1, -1, 4$ — can certainly form a basis w/ eigenvalues

$\Rightarrow$ both similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Ex: Are these diagonizable?

a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Yes! (it already is diagonal)

$\dim(\lambda = 2 \text{ eigenspace}) = 3$

b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = 0$, $x_3 = 0$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\dim(\lambda = 2 \text{ eigenspace}) = 1$

No!

c) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ No, $\dim(\lambda = 2 \text{ eigenspace}) = 2$

<u>Prop:</u> For any λ, A is similar to B

$\iff$ $A - \lambda I$ is similar to $B - \lambda I$

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Pf: $A = CBC^{-1}$

$$A - \lambda I = CBC^{-1} - \lambda I$$

$$A - \lambda I = CBC^{-1} - C\lambda I C^{-1}$$

$$A - \lambda I = C(B - \lambda I)C^{-1} \qquad \square$$

Thm If A and B are similar, they have same characteristic polynomial, eigenvalues, and dimensions of eigenspaces.

Pf. Only left to show char. poly.

Suppose $A = CBC^{-1}$

$$\Rightarrow A - \lambda I = C(B - \lambda I)C^{-1}$$

$$\begin{aligned} \det(A - \lambda I) &= \det(C(B - \lambda I)C^{-1}) \\ &= \det(C)\det(B - \lambda I)\det(C^{-1}) \\ &= \det(B - \lambda I) \end{aligned}$$

$\square$

Ex: Diagonalize (if possible): $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

char. poly: $\begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = 1-\lambda \begin{vmatrix} -5-\lambda & 6 \\ -3 & 4-\lambda \end{vmatrix}$

$= 1-\lambda(-20+5\lambda-4\lambda+\lambda^2+18)$

$= 1-\lambda(\lambda^2+\lambda-2) = (1-\lambda)(\lambda-1)(\lambda+2)$

$\lambda = 1, -2$

e-vecs:

$\lambda = 1$:

$A - I = \left[\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$ $\quad x_1 = x_1, \; x_2 = x_3, \; x_3 = x_3 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -2$

$A + 2I = \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix}$ $\quad x_1 = x_3, \; x_2 = 2x_3, \; x_3 = x_3 \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Yes! Three independent eigen-vectors $\Rightarrow$ diagonizable

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

Defn: Let A be a $n \times n$ matrix w/ eigenvalue λ.

- The <u>algebraic multiplicity</u> of λ is its multiplicity as a root of the characteristic polynomial
- The <u>geometric multiplicity</u> of λ is the dimension of the eigen-space

Fact: geometric multiplicity $\leq$ algebraic multiplicity

Ex:

	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
alg. mult. of $\lambda = 2$:	3	3	3
geom. mult of $\lambda = 2$:	3	1	2

Ex: $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ Find alg. & geom. multiplicities of e-vals. Is A diagonizable?

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} = 2-\lambda \begin{vmatrix} -\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= 2-\lambda(-\lambda+\lambda^2-3)-(\lambda-1-1) = -(\lambda-2)(\lambda^2-\lambda-2)$$

$$= -(\lambda-2)^2(\lambda+1)$$

$$\lambda = 2, -1$$

$\underline{\lambda=2:}$

$$\text{Nul}(A-2I) = \text{Nul}\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & -1 \end{bmatrix}\right) \xrightarrow{R_2+R_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{aligned} x_2 &= 0 \\ x_1 &= x_3 \\ x_3 &= x_3 \end{aligned} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

alg. mult.: 2 geom. mult.: 1

$\underline{\lambda=-1}$

$$\text{Nul}(A+I) = \text{Nul}\left(\begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}\right) \xrightarrow[R_2+R_3]{R_1-3R_3} \begin{bmatrix} 0 & -8 & -6 \\ 0 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \quad \begin{aligned} 4x_2 &= -3x_3 \\ x_1 &= x_2+x_3 \end{aligned} \quad \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

alg. mult.: 1 geom. mult.: 1

Not diagonizable