3/7/25

Thm (recap) Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$, $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ be linear transforms. Then $T = T_2 \circ T_1$ is linear. If A_i is the standard matrix for T_i, then T has the standard matrix $A_2 A_1$

Pf. $\vec{T}(\vec{v}) = \vec{T}_2(\vec{T}_1(\vec{v})) = \vec{T}_2(A_1\vec{v}) = A_2 A_1 \vec{v}$ $\square$

Ex: Let T_α, T_β be rotation by angles α, β. Then $T_\alpha \circ T_\beta$ is rotation by $\alpha + \beta$.

$$T_{\alpha+\beta} \overset{\text{matrix}}{=} \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$T_\alpha \circ T_\beta \overset{\text{matrix}}{=} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

Invertible Functions

Recall: $f: X \to Y$ is invertible if it is bijective

Have $f^{-1}: Y \to X$ w/ $f(x) = y \Leftrightarrow f(y) = x$

$\left(\begin{array}{l} \text{Also: } f(f^{-1}(x)) = x \\ \qquad\quad f^{-1}(f(x)) = x \end{array} \right)$

Thm

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ linear.

① T is invertible $\Longleftrightarrow$ $\ker T = \{\vec{0}\}$ <u>and</u> $m = n$

<u>Pf.</u> Suppose T invertible. Then T is bijective. T inj $\Rightarrow \ker T = \{\vec{0}\}$

T surj $\Rightarrow$ Range of T has $\dim = m$

rk-nullity: $\underbrace{\dim \ker T}_{0} + \underbrace{\dim \text{Range } T}_{m} = n \quad \Rightarrow m = n$

Now suppose $m = n$ and $\ker T = \{\vec{0}\}$. T is inj since $\ker T = \{\vec{0}\}$

rk-nullity: $\underbrace{\dim \ker T}_{0} + \underbrace{\dim \text{Range } T} = n$

$\Rightarrow \dim \text{Range } T = n = m \Rightarrow T$ is surj

② If T is invertible w/ std matrix A,

T^{-1} is invertible w/ std matrix A^{-1}

|

Pf. $T(\vec{v}) = A\vec{v}$, A $n \times n$ w/ $\text{Nul}(A) = \{\vec{0}\}$ $(\ker T = \{0\})$

$\Rightarrow A$ is invertible

$T(\vec{x}) = \vec{y} \iff T^{-1}(\vec{y}) = \vec{x}$

$A\vec{x} = \vec{y} \iff \vec{x} = A^{-1}\vec{y}$ $\Rightarrow T^{-1}(\vec{y}) = A^{-1}(\vec{y})$

$\square$

Cor: $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible $\iff$ its standard matrix A is invertible

Ex: Is $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ invertible? Yes (just rotate by $-\alpha$)

T_α matrix: $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

T_α^{-1} matrix: $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\parallel$

$T_{-\alpha}$

Fact: $\cos(-\alpha) = \cos(\alpha)$
$\sin(-\alpha) = -\sin(\alpha)$

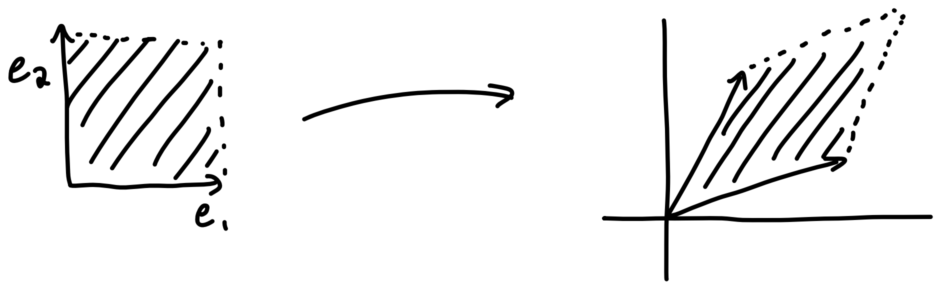
$T_\alpha \circ T_\alpha^{-1} = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Determinants

Idea: Linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$ can distort areas

Want: A number that captures this

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



compare the areas! use a sign to keep track of whether vectors were swapped

We call the determinant of an $n \times n$ matrix A the signed volume of the image of the unit cube, written $\det A$

Basic Principles

- $\det I_n = 1$
- Multiplying a row of A by r multiplies $\det A$ by r.
- Adding a multiple of one row to another row doesn't change $\det A$
- Swapping two rows multiplies $\det A$ by -1

Fact: There is a unique function satisfying all these properties!

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 5R_2} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

Area = 1 (height and base stay constant)