3/5/25

- Quiz : 1) linear transformation defn — 1) $T(u+v) = T(u) + T(v)$
2) $T(c\vec{u}) = cT(\vec{u})$

2) checking if linear transformation is a subspace (of $\mathbb{R}^6$)

method 1: $\boxed{\vec{0}}$ $T(\vec{0}) = 5 \cdot \vec{0}$

$$\begin{matrix} T(\vec{0}) & = & 5\cdot\vec{0} \\ \| & & \| \\ \vec{0} & = & \vec{0} \end{matrix} \checkmark$$

$\boxed{+}$ suppose $T(v) = 5v$, $T(u) = 5u$

$$T(u+v) \stackrel{?}{=} 5(u+v)$$
$$\| \qquad\qquad \|$$
$$T(u) + T(v) \qquad 5$$

$\boxed{*}$ $T(c\vec{v}) = 5(c\vec{v})$

$$\| $$
$$cT(\vec{v}) = c \cdot 5\vec{v}$$

Method 2: Suppose T is rep'd by A :

$$T(\vec{v}) = A\vec{v}$$

<u>want</u> : $\{\vec{v} \mid A\vec{v} = 5\vec{v}\}$

$= \{\vec{v} \mid A\vec{v} - 5\vec{v} = \vec{0}\}$

$= \{\vec{v} \mid (A - 5I)\vec{v} = \vec{0}\} = \mathrm{Nul}(A - 5I)$

! Null spaces are always subspaces

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x+z \end{bmatrix}$$ is T injective?

Recall: T is injective $\iff \ker T = \{\vec{0}\}$

T is rep'd by $A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\rightarrow \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — no free variables

$\Rightarrow \text{Nul}(A) = \{\vec{0}\}$

$\Rightarrow \ker T = \{\vec{0}\}$

$\Rightarrow T$ is injective

Thm (Rk Thm)

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear

Rk-null for A:

$$\text{rk} A + \dim(\text{Nul}(A)) = n$$

Then $\dim \text{Range}(T) + \dim \ker(T) = n$

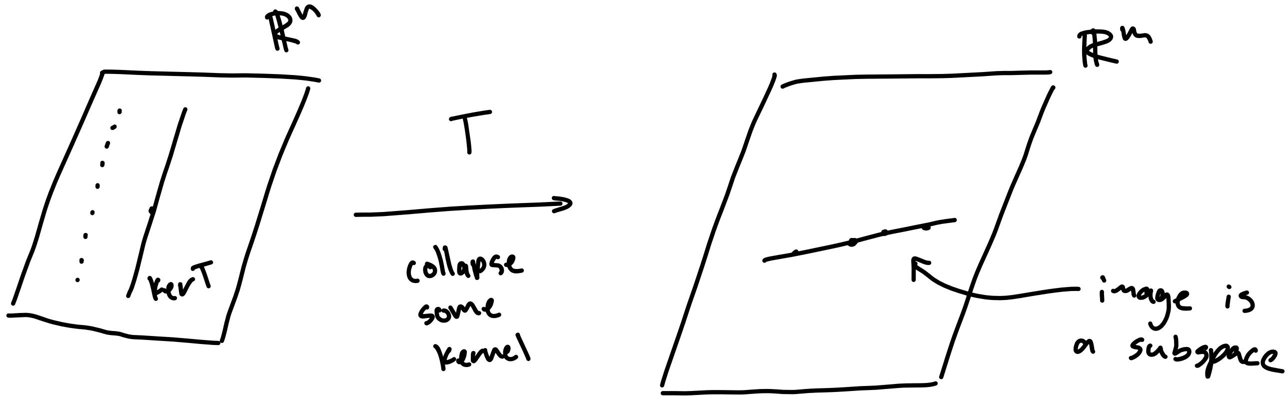
Pf. Let $\underbrace{A}_{m \times n}$ represent T

rk-nullity $\Rightarrow \text{rk} A + \dim \text{Nul} A = n$

$\Rightarrow \dim(\text{col} A) + \dim(\text{Nul} A) = n$

$\Rightarrow \dim(\text{Range} T) + \dim(\ker T) = n$ $\square$

Picture:



(continued from prev)

Ex:

Is $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x+z \end{bmatrix}$ surjective?

$$\dim \operatorname{Range}(T) + \underbrace{\dim \ker(T)}_{0} = 3$$

$$\dim \operatorname{Range}(T) = 3$$

So T is surjective ($\Rightarrow$ T is bijective)

Composition

Thm Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations.

The comp $T = T_2 \circ T_1$ is linear.

If A_i is the standard matrix for T_i, the matrix for T is $A_2 A_1$.