3/31/25

(Review)

<u>Ex</u>: Find the e-vals and e-vecs of $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

① to find e-vals: find roots of characteristic polynomial

$\det(A - \lambda I) = 0$, solns λ are e-vals

② to find e-vecs: find null space of $A - \lambda I$

$Nul(A - \lambda I)$

<u>Pf review</u>

$A\vec{v} = \lambda \vec{v}$

$\Leftrightarrow A\vec{v} - \lambda\vec{v} = 0$

$\Leftrightarrow (A - \lambda I)\vec{v} = 0$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{bmatrix}$$

①

$$\det(A - \lambda I) = (2-\lambda)\begin{vmatrix} -\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - \lambda - 3) - (\lambda - 1 - 1)$$

$$= -(\lambda - 2)(\lambda^2 - \lambda - \;) - (\lambda - 2)$$

$$= (\lambda - 2)(\lambda^2 - \lambda - 2)$$

$$= (\lambda - 2)(\lambda - 2)(\lambda + 1)$$

$$\lambda = -1, 2$$

(2)

$$\text{Nul}(A-\lambda I) \overset{\lambda=2}{=} \text{Nul}\left(\begin{bmatrix} 2-2 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & 1-2 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & -1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix}$$

let $x_3 = 1$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A-\lambda I) \overset{\lambda=-1}{=} \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_3 = x_3 \qquad x_1 = -2x_3 - 3x_2$$

$$x_2 = \frac{-3x_3}{4} \qquad = -2x_3 + \frac{9}{4}x_3 = \frac{1}{4}x_3$$

$$\vec{v} = x_3 \begin{bmatrix} 1/4 \\ -3/4 \\ 1 \end{bmatrix}$$

"eigen space"

let $x_3 = 4$

$$\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ e-vals = 1

e-vecs = anything (except $\vec{0}$ by def)

$$0 = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

$$\text{Nul}(A-\lambda I) \overset{\lambda=1}{=} \text{Nul}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathbb{R}^2$$

Defn: For an eigenvalue λ of A, the λ-eigenspace of A is $\mathrm{Nul}(A-\lambda I)$

Ex: Find e-vals, e-vecs of

a) $A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \det(A_1 - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 = 0$
$\Rightarrow \lambda = 2$

$\mathrm{Nul}(A_1 - 2I) = \mathrm{Nul}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \Rightarrow$ eigenspace $= \mathbb{R}^3$

b) $A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \det(A_2 - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)^3 = 0 \Rightarrow \lambda = 2$

$\mathrm{Nul}(A_2 - 2I) = \mathrm{Nul}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$ $\quad x_2 = 0$

e-space $= \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

dim $= 2$

c) $A_3 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad \det(A_3 - \lambda I) = (2-\lambda)^3 = 0 \Rightarrow \lambda = 2$

$\mathrm{Nul}(A_3 - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = 0$, $x_3 = x_3$ $\quad$ e-space $= \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Ex: $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ 1) what does $T_A(\vec{v}): \mathbb{R}^2 \to \mathbb{R}^2$ look like?

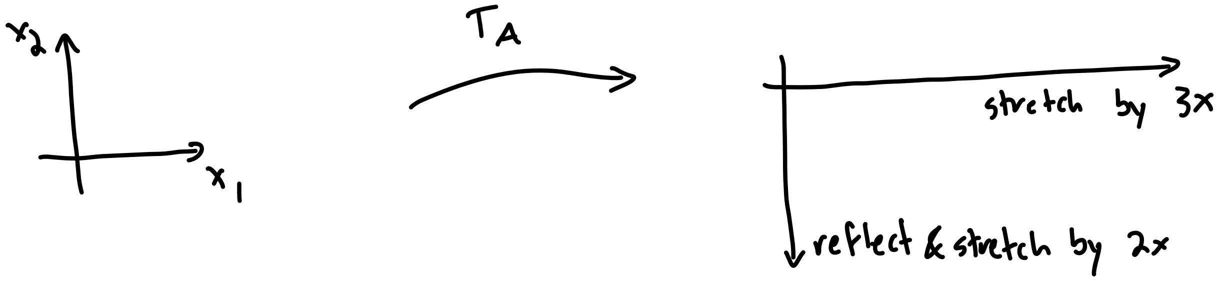
2) what is A^{10}?

$$A^{10}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3^{10}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3^{10} \\ 0 \end{bmatrix}$$

$$A^{10}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-2)^{10}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2)^{10} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ -2x_2 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 3^{10} & 0 \\ 0 & (-2)^{10} \end{bmatrix}$$



<u>upshot</u>: diagonal matrices make it easy to find powers of A