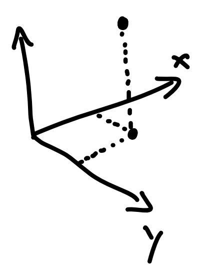
3/3/25

- Reflection due friday, HW 6 due mon after break

Def^n: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

- The <u>kernel</u> of T, written as $\ker T$, is $\{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) = \vec{0}\}$ ~ kernel of matrix transformation is null space
- If $S \subseteq \mathbb{R}^n$ is a subset, the <u>image</u> of S under T, written $T[S]$, is $\{T(\vec{v}) \mid \vec{v} \in S\}$. ~ output space
- If $S \subseteq \mathbb{R}^m$ is a subset, the <u>preimage</u> of S, written $T^{-1}[S]$, is $\{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) \in S\}$ ~ input space
- The <u>range</u> of T is $\text{range}(T) = T[\mathbb{R}^n]$

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^2 \quad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ - projection onto xy-plane



all things that map to origin

$\ker T = \left\{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R}\right\}$ - z-axis

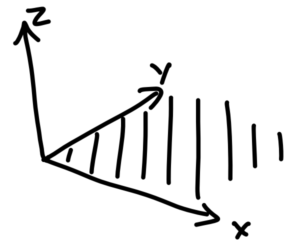
$= T^{-1}[\{0\}]$

$\text{range } T = \mathbb{R}^2$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

— thinking of xy-plane as lying in $\mathbb{R}^2$



$$\ker T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{range } T = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$ — xy-plane

Defn: A function $f: X \to Y$ is:

domain (X), codomain (Y) — think type — what is allowed to be mapped to

~ range: what is actually mapped to

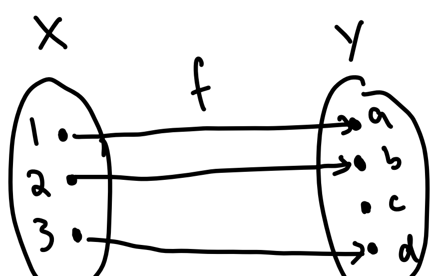
- injective (or one-to-one) if whenever $f(a) = f(b)$, then $a = b$ ~unique y for each x

- surjective (or onto) if for every $b \in Y$, there is $a \in X$ w/ $f(a) = b$

Rmk: f is surjective $\iff$ range$(f) = Y$

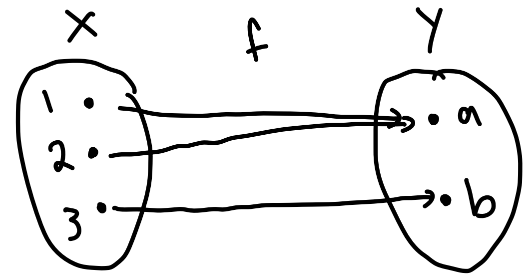
- bijective (or one-to-one correspondence) if it is both injective and surjective.

Ex:



injective but not surjective

Ex:



is surjective but not injective since $f(1) = a = f(2)$ but $1 \neq 2$

Ex: Projection onto xy-plane is surjective but not injective

Ex: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is injective but not surjective

show inj.: $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = c \\ b = d \end{matrix} \quad \checkmark$$

show not surj.: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never be $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

☆ can't be surjective if mapping to higher dimension!

Thm $T : \mathbb{R}^m \to \mathbb{R}^n$ linear.

1) If $V \subseteq \mathbb{R}^n$ is subspace, $T[V] \subseteq \mathbb{R}^m$ is a subspace
2) If $W \subseteq \mathbb{R}^m$ is a subspace, $T^{-1}[W]$ is a subspace
3) $\ker T$ is a subspace
4) T is injective $\iff \ker T = \{\vec{0}\}$

Pf. 1), 2) omitted

3) $\ker T = T^{-1}[\{\vec{0}\}]$

subspace

so, $\ker T$ is also a subspace — from 2)

4) Suppose T injective. Let $\vec{v} \in \ker T$. Then

$T(\vec{v}) = \vec{0} = T(\vec{0})$, so $\vec{v} = \vec{0}$

Now suppose $\ker T = \{\vec{0}\}$. Suppose $T(\vec{a}) = T(\vec{b})$

$$T(\vec{a}) - T(\vec{b}) = \vec{0}$$
$$\parallel$$
$$T(\vec{a} - \vec{b}) = \vec{0}$$

$\longrightarrow$ $\vec{a} - \vec{b} \in \ker T$

$\vec{a} - \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{b}$ $\square$

Rmk: Suppose A matrix representing $T: \mathbb{R}^n \to \mathbb{R}^m$. Then:

- $\text{Range}(T) = \text{Col}(A) = \{ \overset{T(\vec{x})}{\overset{\parallel}{A\vec{x}}} \mid \vec{x} \in \mathbb{R}^n \}$

- $\ker T = \text{Nul}(A) = \{ \vec{x} \mid \underset{T(\vec{x})}{\underset{\parallel}{A\vec{x}}} = 0 \}$