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## Eigenvectors

Defn: $A$ $n \times n$ matrix, $\lambda$ is an eigenvalue for $A$ if there is a nonzero $\vec{v} \in \mathbb{R}^n$ w/ $A\vec{v} = \lambda \vec{v}$.

$\vec{v}$ is called an eigenvector of $A$ corresponding to $\lambda$

Ex: $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ which are e-vecs of $A$?

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{A \times} \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ No

b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{A \times} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Yes, $\lambda = -1$

c) $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \xrightarrow{A \times} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ No

d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{A \times} \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ Yes, $\lambda = 5$

e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{A \times} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ No (not nonzero)

Prop If $\vec{v}$ is an e-vec of $A$ for any $c \neq 0$, $c\vec{v}$ is an e-vec also (& w/ same e-val)

Pf know: $A\vec{v} = \lambda \vec{v} \Rightarrow A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$

Ex: a) Does $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ have an e-vec w/ $\lambda = 3$?

b) Does $B = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$ have an e-vec w/ $\lambda = -3$?

Soln: $A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = 0$

$\iff A\vec{v} - \lambda I_n\vec{v} = 0 \iff (A - \lambda I_n)\vec{v} = 0$

a) $(A - 3I)\vec{v} = 0$

$A - 3I = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ det = 1 No.

$\Rightarrow$ only such $\vec{v}$ is $\vec{v} = 0$

b) $(B + 3I)\vec{v} = 0$

$B + 3I = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$ det = 0 $(\text{Nul } B \neq \{0\})$

Yes! bc...

ex: $\vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$3x_1 + 2x_2 = 0$

$x_1 = -\frac{2}{3}x_2$ free var

let $x_2 = 3 \Rightarrow x_1 = -2$

check: $\begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$

Thm)

1. The eigenvalues of $A$ are the roots of the polynomial $\det(A\lambda - I)$.

Pf $\lambda$ is an e-val $\Leftrightarrow$ there is a non zero $\vec{v}$ with

$$A\vec{v} = \lambda\vec{v} \iff \overset{\text{"there exists"}}{\exists}\ \vec{v} \neq \vec{0} \text{ w/ } (A - \lambda I)\vec{v} = \vec{0}$$

$$\iff \text{Nul}(A - \lambda I) \neq \{\vec{0}\} \iff \det(A - \lambda I) = 0$$

$\square$

Defn: The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A

② The set of eigenvectors for A w/ e-val $\lambda$ is

$$\text{Nul}(A - \lambda I) \setminus \{\vec{0}\}$$

Pf. By the above ①,

$$A\vec{v} = \lambda\vec{v} \iff \vec{v} \in \text{Nul}(A - \lambda I).$$

$\square$

Ex: Find the e-vals & e-vecs of $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{bmatrix} \qquad \det(A - \lambda I) = -\lambda(3-\lambda) - 4$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda - 4)(\lambda + 1) = 0$$

e-vals: $\lambda = 4, -1$

(Ex contd.)

$\underline{\lambda = 4}$ $\quad A - \lambda I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$

$$-x_1 = -2x_2$$
$$x_1 = 2x_2$$
$$x_2 = x_2$$

(Finding vector in Nul $(A - \lambda I)$)

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda = -1}$ $\quad A - \lambda I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

$$2x_1 + x_2 = 0$$
$$2x_1 = -x_2$$
$$x_2 = x_2$$

$$\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

check: $A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

## Challenge Problem!

$A$ $2 \times 2$. Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be an e-vec w/ e-val $-1$, let $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be an e-vec w/ e-val $2$. Compute $A^5$