3/24/25

Review for Exam 2:

Vectors / Linear Eqns:

- Subspaces — Def: $S \subseteq \mathbb{R}^n$ is a subspace if:

  Ex: Nul (A)
  - $0 \in S$

  Col (A)
  - If $\vec{v}, \vec{u} \in S$, then $u+v \in S$

  Span$\{\vec{v}_1, \dots, \vec{v}_k\}$
  - If $\vec{u} \in S$, $c \in \mathbb{R}$, then $c\vec{u} \in S$

- Basis — Def: $\{v_1, \dots v_k\}$ is a basis for a subspace S if:
  - $\{v_1, \dots, v_k\}$ are lin. ind.

  (coordinates w.r.t. basis)
  - $\{v_1, \dots, v_k\}$ span S

- Dimension — Def: The number of vectors in a basis of a subspace S is the dimension of S

Ella's point: If S is k-dimensional then k vectors in S are lin. ind. $\Longleftrightarrow$ they span S

(Special case:) If $S = \mathbb{R}^n$, n vectors in $\mathbb{R}^n$ are lin ind $\Longleftrightarrow$ they span $\mathbb{R}^n$

- Homogenous — solutions to system are Null space

- Rank-nullity thm — $\dim \text{Nul} A + \overbrace{\dim \text{Col} A}^{\text{rank } A} = n$

- Solutions to $A\vec{x} = b$ are a translate of Nul A

  ↳ have the form $\vec{n} + \vec{p}$ where $\vec{p}$ is fixed soln $A\vec{p} = \vec{b}$ and $\vec{n}$ varies over all elements of Nul A

- Abstract vector spaces (polynomials)

Linear Transformations

Def: $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

- for $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
- for $\vec{u} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$

- How to write them, basic examples
- standard matrix — $A = [\,T(e_1)\ T(e_2)\ \cdots\ T(e_n)\,]$
  - example problems finding transformations / standard matrix from starting info
- injective, surjective, bijective
  ($\ker T = \{0\}$) To be invertible
- kernel, image
- rank-nullity ($\dim \ker T + \dim \operatorname{Range} T = n$)
- equivalent conditions for $n \times n$ matrix to be invertible:
  (many) - $\det A \neq 0$ - cols are ind - pivot in every column
  - RREF $= I_n$ - $\operatorname{col} A = \mathbb{R}^n$ - $\operatorname{nul} A = \{0\}$

Determinants

- How to compute w/ row reduction
- How to compute w/ cofactor expansion

- $\det A \neq 0 \iff A$ is invertible

Practice Test

Bonus #2) AB is invertible, show A, B is invertible

- know cols AB span $\mathbb{R}^n$

  $\Rightarrow$ can find vectors $AB\vec{v}_1, \dots, AB\vec{v}_n$ spanning $\mathbb{R}^n$

  $A\vec{u}_1, \dots, A\vec{u}_n$ spans $\mathbb{R}^n$ for $\vec{u}_i = B\vec{v}_i$

  $\Rightarrow \operatorname{Col} A = \mathbb{R}^n \Rightarrow A$ invertible

- if B is not invertible, there is $\vec{v} \neq \vec{0}$, $v \in \operatorname{Nul} B$

  $\Rightarrow \quad B\vec{v} = 0$

  $AB\vec{v} = 0$

  $\Rightarrow \vec{v} \in \operatorname{Nul}(AB) = \{\vec{0}\}$ $\rightarrow\leftarrow$

  $\square$

Rmk

$f \circ g$ invertible

$\Rightarrow f$ is surj, g inj