3/21/25

- office hours M 11-12, 3-4

Ex:

$$\begin{vmatrix} -2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 4 & 2 & -3 \end{vmatrix} = -2 \begin{vmatrix} 4 & 2 & 0 \\ 8 & 4 & 0 \\ 4 & 2 & -3 \end{vmatrix}$$

$$= -2(3) \begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = -6(16-16) = 0$$

alt: $R_3 - 2R_2$ (no change to det)

$$\begin{vmatrix} -2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & 3 \end{vmatrix} = 0$$

(rows are dep, too)

Fact: (recap)
- $\det A^T = \det A$
- $\det(AB) = \det A \det B$

Cor $\det A^{-1} = \dfrac{1}{\det A}$ if A invertible

Pf. $AA^{-1} = I_n \Rightarrow \det(AA^{-1}) = \det I_n$

$\Rightarrow \det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \dfrac{1}{\det A}$ $\square$

Thm Let A $n \times n$ invertible w/ cofactors C_{ij}.

Then, $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & \vdots \\ C_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \\ & \cdots & & C_{nn} \end{bmatrix}$ flipped from usual index

Defn: The matrix $\begin{bmatrix} C_{11} & C_{21} & \cdots \\ C_{12} & \ddots & \vdots \\ \vdots & \cdots & C_{nn} \end{bmatrix}$ is called the

<u>adjugate</u> of A

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \operatorname{adj} A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$C_{11} = 4(-1)^{1+1} = 4$

$C_{21} = 2(-1)^{1+2} = -2$

$C_{12} = 3(-1)^{1+2} = -3$

$C_{22} = 1(-1)^{2+2} = 1$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \det A \cdot I_n$$

Rmk: We always have $A \cdot \operatorname{adj} A = \det A \cdot I_n$

Ex: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, find A^{-1}

$C_{11} = d$

$C_{21} = -b$

$C_{12} = -c$

$C_{22} = a$

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ compute A^{-1} using cofactors

$C_{11} = (-1)^{1+1}(-1) = -1$

$C_{21} = (-1)^{1+2}(-1) = 1$

$C_{31} = (-1)^{1+3}(-1) = -1$

$C_{12} = (-1)^{1+2}(-1) = 1$

$C_{22} = (-1)^{2+2}(-1) = -1$

$C_{32} = (-1)^{3+2}(1) = -1$

$C_{13} = (-1)^{1+3}(-1) = -1$

$C_{23} = (-1)^{2+3}(1) = -1$

$C_{33} = (-1)^{3+3}(1) = 1$

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A \cdot \text{adj} A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Thm (Cramer's Rule)

Let A be $m \times n$ and invertible. Let $\vec{b} \in \mathbb{R}^n$

Let A_i be matrix replacing i^{th} column of A w/ $\vec{b}$.

Then $x_1 = \dfrac{\det A_1}{\det A}$, $x_2 = \dfrac{\det A_2}{\det A}$, ..., $x_n = \dfrac{\det A_n}{\det A}$

is the unique solution of $A\vec{x} = \vec{b}$

Ex: Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$A_1 = \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}$

$$x_1 = \frac{\det A_1}{\det A} = \frac{20-12}{4-6} = \frac{8}{-2} = -4$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{6-15}{4-6} = \frac{-9}{-2} = \frac{9}{2}$$

Check: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Defn: Let A $n \times n$. A number λ is an <u>eigenvalue</u> for A if there is a vector $\vec{v} \in \mathbb{R}^n$ w/ $A\vec{v} = \lambda\vec{v}$. $\vec{v}$ is called an <u>eigenvector</u> corresponding to λ

Ex: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A

$$A\vec{v} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda = 4$$

(so is any scalar mult. of eigenvector)

$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is also an eigenvector:

$$A\begin{bmatrix} 3 \\ 3 \end{bmatrix} = A \cdot 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \lambda = 4$$

Questions:
- how do we find the various λ?
- how do we find corresponding e-vectors?
- what do they tell us about the matrix A?