3/17/25

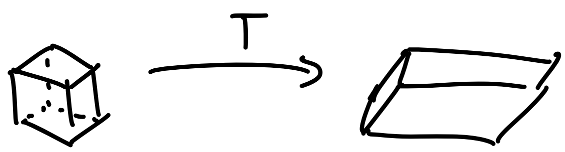
Exam 2 next wednesday (3/26)
HW 7 due monday (3/24)

Determinants Contd.

Idea: $T: \mathbb{R}^n \to \mathbb{R}^n$

How much does T stretch volumes by?

$\det T = \text{vol}(T(\text{unit cube}))$ w/ sign



Basic Principles:

- $\det I_n = \underline{1}$
- swapping rows changes det by $*(-1)$
- scalar multiplying a row by $r \neq 0$ multiplies det by r
- Adding mult of one row to another doesn't change det

Ex: $\det[7] = 7\det[1] = 7 * 1 = 7$

Ex: $\det[a] = a$

Ex: $\det\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix} = 5\det\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix} = 5\cdot 3\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 15$

Ex: $\det\begin{bmatrix}a_1 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & a_n\end{bmatrix} = a_1 \cdot a_2 \cdot \dots \cdot a_n$

Ex: $\det\begin{bmatrix}5 & 6\\ 0 & 3\end{bmatrix} \underset{\substack{\uparrow\\ R_1 - 2R_2 \text{ (no change to det)}}}{=} \det\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix} = 15$

Ex: $\det\begin{bmatrix}a_1 & * & * & *\\ 0 & \ddots & * & *\\ 0 & 0 & \ddots & *\\ 0 & 0 & & a_n\end{bmatrix} = a_1 \cdots a_n$

Defn: $A = [a_{ij}]$ is <u>upper triangular</u> if $a_{ij} = 0$ when $i > j$

Ex: $\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \underset{\substack{\uparrow\\ R_2 - 3R_1}}{=} \begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix} = -2$

general formula!

Ex/Prop: $\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = ?$

Pf. <u>if $a \neq 0$</u> $\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} \underset{\substack{\uparrow\\ R_2 - \frac{c}{a}R_1}}{=} \begin{bmatrix}a & b\\ 0 & d - \frac{bc}{a}\end{bmatrix} = a\left(d - \frac{bc}{a}\right)$

$= ad - bc$

(upper triangular)

<u>If $a=0$</u> $\det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{=} -\det\begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = -bc$

$= \underbrace{ad}_{=0} - bc$

(squashed cube! zero volume)

<u>Ex:</u>

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} = 0 \cdot \det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

<u>Rmk:</u> If A has a row of all zeros, $\det A = 0$.

<u>Prop</u>

$\det A = 0 \iff$ cols of A are linearly dependent

<u>Pf.</u>

To compute $\det A$, perform row reduction on A

swapping rows, multiplying by nonzero scalars, and taking linear combos of rows preserve whether or not $\det A = 0$

So, $\det A = 0 \iff \det RREF(A) = 0$

$\iff RREF(A) \neq I_n \iff$ cols of A are dep.

$\square$

Defn: Let A be an $m \times n$ matrix.

- For $1 \leq i, j \leq n$, the <u>(i,j)-minor</u> of A, written A_{ij}, is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column
- The <u>(i,j)-cofactor</u> of A, usually called C_{ij}, is $C_{ij} = (-1)^{i+j} \det A_{ij}$

Ex: $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 1 \end{bmatrix}$

delete 2nd row, delete 1st column

$A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C_{21} = (-1)^{2+1} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$

Useful Fact: For $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \vdots \\ a_{21} & & \ddots & \vdots \\ & \cdots & & a_{nn} \end{bmatrix}$

$$\det A = a_{i1} C_{i1} + \dots + a_{in} C_{in}$$

for any $1 \leq i \leq n$

Ex: Find $\det\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 1 \end{bmatrix}$

$$= 4\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 4 - 0 + 1 = 5$$

$$\det\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 1 \end{bmatrix} = -3\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= -3 + 4 - (-4) = 5$$