2/7/25

- Exam 1 content up until (including) elementary matrices

Recall: A $n \times n$, A^{-1} $n \times n$, $A^{-1}A = I = A^{-1}A$

useful:

$A\vec{x} = \vec{b}$, if a is invertible

$\hookrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow x = A^{-1}\vec{b}$ ← is a solution

$A^{-1}A\vec{x} = I\vec{x}$

Prop: If A $n \times n$ w/ $AB = I$, then A can be row reduced to I.

Pf. Since $AB = I$, $A\vec{x} = \vec{b}$ always has the solution $\vec{x} = B\vec{b}$. Thus, A must have a pivot in every row/col $\Rightarrow$ A row reduces to I

$\square$

Check: $\vec{x} = B\vec{b}$

$A\vec{x} = \vec{b}$ plug in

$AB\vec{b} = \vec{b}$ ✓

$I\vec{b} = \vec{b}$

Thm If A and B are $n \times n$, then

$$AB = I \iff BA = I$$

Pf. Suppose $AB = I$. Then A can be row reduced to I, so there is a matrix C (a product of elementary matrices) w/ $CA = I$

$$B = IB = CAB = CI = C$$

so $B = C$, so $BA = I$ $\square$

Corollary: If A is $n \times n$, $AB = I \iff A$ is invertible with $B = A^{-1}$

Example: Every elementary matrix is invertible with an inverse that is also an elementary matrix

(undo actions)

$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has inverse $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

$R_1 \leftrightarrow R_2$

$E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has inverse $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\frac{1}{3}R_1$

$3R_1$

$E_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has inverse $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 - 4R_3$

$R_1 + 4R_3$

Thm $(AB)^{-1} = B^{-1}A^{-1}$

Pf: $(AB)(B^{-1}A^{-1})$

(associative) $= A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$. $\square$

To find A^{-1}

Find a matrix B whose columns are solutions to $A\vec{x} = e_i$. If such a matrix exists, then A is invertible with $A^{-1} = B$. Otherwise, A is not invertible.

Pf. $A \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix}$

If $A\vec{v}_i = \vec{e}_i$ for all i, then $A[v_1 \cdots v_n] = I$.

Otherwise, $A\vec{x} = e_i$ does not have a solution for some i, so A does not have a pivot in every column, so not invertible.

Practically Speaking: Given A,

① put in form $[A|I]$

② row reduce to $[I|A^{-1}]$ (if possible; if not, A is not invertible)

Ex: $A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$ Find A^{-1}

Soln.
$$\left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[-2R_2]{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 7/2 & 1/2 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - \frac{7}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 7 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$$

Thm The following are equivalent for an $n \times n$ matrix A:

i) A is invertible

ii) $RREF(A) = I$

iii) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$

iv) A is a product of elementary matrices

v) The span of the columns of A is $\mathbb{R}^n$