2/3/25

Recap: $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $\qquad AI_n = A = I_m A$

$A\vec{x} = b$, $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

$$\begin{array}{c} A\vec{x} \\ \| \\ x_1\vec{v}_1 + \dots + x_n\vec{v}_n \\ \vdots \end{array}$$

Other Matrix Operations:

- Addition: $A = [a_{ij}]$, $B = [b_{ij}]$ are $m \times n$ matrices

  $A + B$ is matrix with entries $[a_{ij} + b_{ij}]$

- Scalar Multiplication: $rA = [ra_{ij}]$

Ex: $A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -5 & 3 \end{bmatrix}$

$$2A - 3B = \begin{bmatrix} 2 & 4 & -8 \\ 0 & 6 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -6 \\ -3 & 15 & -9 \end{bmatrix} = \begin{bmatrix} 5 & 4 & -14 \\ -3 & 21 & -11 \end{bmatrix}$$

Zero Matrix:

0 is the matrix of all zeros

(size is often clear from context)

$$0 + A = 0 + A = 0$$

Def^n: The <u>transpose</u> of an $n \times m$ matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$

Def^n: A square matrix is called <u>symmetric</u> if $A^T = A$

Ex: Find A^T when $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & 7 \end{bmatrix}$

Soln: $A^T = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ 5 & 7 \end{bmatrix}$

Ex: Fill in the missing entries of

think "flipping" about matrix across the diagonal

$$\begin{bmatrix} 5 & -6 & \boxed{-2} & 8 \\ \boxed{-6} & 3 & \boxed{1} & \boxed{11} \\ -2 & 1 & 0 & 4 \\ \boxed{8} & 11 & 4 & -1 \end{bmatrix}$$

to make it symmetric.

Transpose (A^T)

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

$m \times n$ $\quad$ $n \times p$ $\quad$ $p \times n$ $\quad$ $n \times m$

$\to m \times p$ $\quad$ $\to p \times m$

Pf. If $A = [a_{ij}]$ $B = [b_{ij}]$

$$AB = \left[\sum_{l=1}^{n} a_{il} b_{lj} \right]$$

$$(AB)^T = \left[\sum_{l=1}^{n} a_{jl} b_{li} \right]$$

$$= \left[\sum_{l=1}^{n} b_{li} a_{jl} \right] = B^T A^T \quad \square$$

Non-property: $AB \neq BA$ in general

also, $AB = AC \not\Rightarrow B = C$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 18 & 24 \end{bmatrix}$$

(loses information
~ a is not invertable)

Elementary Matrices

Defn: A matrix that can be obtained from an identity matrix by doing a single elementary row operation (① swap, ② add a mult. of one row to another, ③ scaling) is called an elementary matrix.

Ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3R_2$: $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_1 \leftrightarrow R_2$: $L_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_3 - 7R_1$: $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 7 & 8 & 9 \end{bmatrix}$

$\hookrightarrow L_2 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

$$L_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -21 & -27 & -33 \end{bmatrix}$$

(oops, this shows R3 - 7R*2*)

Observation/Theorem:

Multiplying a matrix A on the left by an elementary matrix L is equivalent to performing the corresponding elementary operation on A.

Pf. Recall that if $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$, then

$$LA = \begin{bmatrix} | & & | \\ Lv_1 & \cdots & Lv_n \\ | & & | \end{bmatrix},$$ so it is enough to show this for products Lv, w/ $\vec{v}$ a vector.

We show this for L corresponding to $R_k - rR_\ell$

$$\begin{array}{c} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ -r & & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_k - rv_\ell \\ \vdots \\ v_n \end{bmatrix} \\ \uparrow \\ k,\ell \end{array}$$

$\square$