2/28/24

Thm

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$B = \{b_1, \dots, b_n\}$ basis for $\mathbb{R}^n$

For $\vec{v} \in \mathbb{R}^n$, $T(\vec{v})$ is completely determined by $T(b_1), \dots, T(b_n)$

Pf:

$$\vec{v} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$$

$$\begin{aligned} T(\vec{v}) &= T(c_1\vec{b} + \dots + c_n\vec{b}_n) \\ &= T(c_1\vec{b}) + \dots + T(c_n\vec{b}_n) \\ &= c_1 T(\vec{b}_1) + \dots + c_n T(\vec{b}_n) \end{aligned}$$

$\square$

Ex:

$T : \mathbb{R}^2 \to \mathbb{R}^3$

(!) linearly independent

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

what is $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$?

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Thm

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$m \times n$

$$A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$$

NB: e_i are the standard basis for $\mathbb{R}^n$,

A is the standard matrix representation of T

Then, $T(\vec{x}) = A\vec{x}$

Pf.

T is determined by $T(e_1), \dots, T(e_n)$

$$\vec{X} = c_1\vec{e}_1 + \dots + c_n\vec{e}_n = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$T(\vec{x}) = T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 T(e_1) + \dots + c_n T(e_n)$$

$$= A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A\vec{x}$$

Ex: $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ $T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(\vec{u}) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $T(\vec{v}) = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$ Find the standard A

1. write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combos of $\vec{u}, \vec{v}$

$$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|cc} I_2 & 5 & 3 \\ & 2 & 1 \end{array}\right]$$

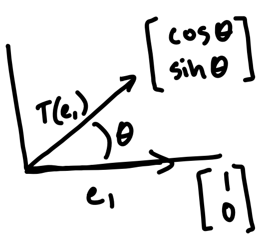
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5\vec{u} + 2\vec{v}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\vec{u} + \vec{v}$$

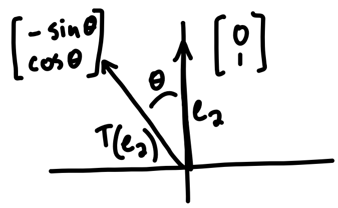
$$\begin{aligned} T(e_1) &= T(5\vec{u}) + T(2\vec{v}) \\ &= 5T(\vec{u}) + 2T(\vec{v}) \\ &= \begin{bmatrix} -10 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 \\ -14 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} T(e_2) &= T(3\vec{u}) + T(\vec{v}) \\ &= 3T(\vec{u}) + T(\vec{v}) \\ &= \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 0 & -1 \\ -9 & 4 \\ 2 & 1 \end{bmatrix}$$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation by counterclockwise θ





$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$

Find the S.M.R for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - y \\ y \\ x \end{bmatrix}$

$$A = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or...

$$\begin{bmatrix} 3x-y \\ y \\ x \end{bmatrix} = \begin{bmatrix} 3x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$$