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Quiz recap: Rank is Dim(Col(A))

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ r \end{bmatrix}$ basis $\mathbb{R}^2$ whenever $r \neq -2$

Solns to $A\vec{x} = \vec{b}$ look like translate of Nul(A) by single soln.

<u>Review:</u>

$T: R^n \to R^m$ is a linear transformation

if preserves sums, i.e. $T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}, \in \mathbb{R}^n$

and preserves scalar mult., i.e. $T(c\vec{u}) = cT(\vec{u})$ for all $c \in \mathbb{R}$, $\vec{u} \in \mathbb{R}^n$

<u>Key Example</u>

Let A be an $m \times n$ matrix

Then $T(x) = A\vec{x}$ is a linear transformation

1) $T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$

2) $T(c\vec{u}) = Ac\vec{u} = cA\vec{u} = cT(\vec{u})$

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x-y \\ x \\ y \end{bmatrix}$$ is a linear transformation.

check: ① $$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right) = \begin{bmatrix} 3(x+w)-(y+z) \\ x+w \\ y+z \end{bmatrix}$$

$$= \begin{bmatrix} 3x-y \\ x \\ y \end{bmatrix} + \begin{bmatrix} 3w-z \\ w \\ z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right)$$

② $$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} 3cx-cy \\ cx \\ cy \end{bmatrix} = c\begin{bmatrix} 3x-y \\ x \\ y \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ is a translation.

Not linear! $$T(\vec{0}) = \vec{0} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \vec{0}$$

Ex: $$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} |x| \\ y \end{bmatrix} \quad T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \end{bmatrix} \quad T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+1 \\ x-2y \end{bmatrix}$$

↳ not linear! (T_1)

$$T_1\left(-2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$-2T_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \quad \neq$$

↳ not linear! (T_2)

$$T_2\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$2T_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

or...

$$T_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T_2\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$T_3 \to$ not linear! $T_3\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \neq \vec{0}$

<u>Ex:</u> $T: \mathbb{R} \to \mathbb{R}$

$T(x) = x^2$ Not linear! (phew!)

$T(2 \cdot 1) = 4 \neq 2T(1) = 2$

<u>Ex:</u> What are the linear maps $T: \mathbb{R} \to \mathbb{R}$?

$T(x) = cx$ are the only ones:

$T(x) = T(x \cdot 1) = x\,T(1)$ so $T(x)$ is totally determined by $T(1)$, and $T(x) = cx$ where $c = T(1)$.