2/21/25

Quiz recap (from last class):

7. If $AC = B$ and two of the matrices are invertible, then so is the third. <u>True!</u>

<u>Pf.</u> A, C inv $\Rightarrow B^{-1} = (AC)^{-1} = C^{-1}A^{-1}$

A, B inv $\Rightarrow AC = B$

$$A^{-1}AC = A^{-1}B$$

$$C = A^{-1}B$$

product of two invertible matrices, also invertible!

$$C^{-1} = B^{-1}A$$

$C, B,$ inv $\Rightarrow AC = B$

$$ACC^{-1} = BC^{-1}$$

$$A = BC^{-1} \qquad A^{-1} = CB^{-1}$$

8. If 7 vectors in $\mathbb{R}^8$ are lin. ind. they span $\mathbb{R}^8$.

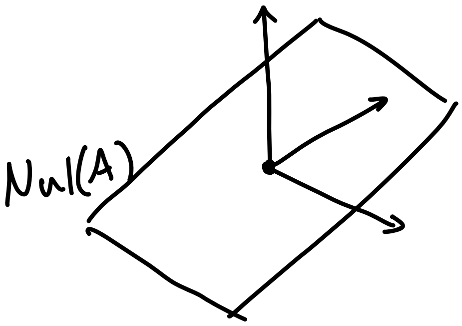
<u>False</u> (at most can span $\mathbb{R}^7$)

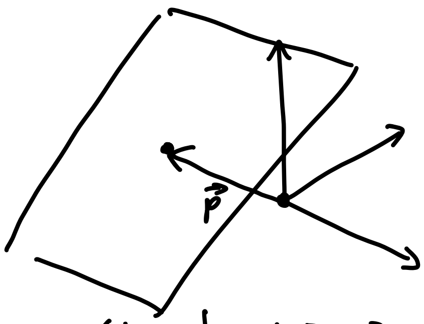
9. If 3 vectors in $\mathbb{R}^3$ are lin. ind. they span $\mathbb{R}^3$

<u>True</u>

aside: "If you're already independent, you will be the basis of whatever you span"

Thm (repeat) Let $A\vec{x} = \vec{b}$ be a consistent system w/ one solution $\vec{p}$. Then all solutions have form $\vec{p} + \vec{n}$ for $n \in \text{Nul}(A)$





Solns to $A\vec{x} = \vec{b}$

Ex: Express solutions to

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 2 & -3 & 4 & -3 & -1 \\ 3 & -5 & 5 & -4 & 3 \\ -1 & 1 & -3 & 2 & 5 \end{array}\right] \end{array}$$

as $\vec{x} = \vec{p} + \underbrace{t_1\vec{v}_1 + \ldots + t_k\vec{v}_k}_{\text{Nul}(A)}$

where $\vec{v}_1, \ldots, \vec{v}_k$ is a basis for solutions to homogenous system

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} \underline{1} & 0 & 5 & -3 & -14 \\ 0 & \underline{1} & 2 & -1 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivots $\quad$ free vars

$x_1 = -5x_3 + 3x_4 - 14$

$x_2 = -2x_3 + x_4 - 9$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5x_3 + 3x_4 - 14 \\ -2x_3 + x_4 - 9 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -14 \\ -9 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$$

Proof of thm: we know $A\vec{p} = \vec{b}$. For $\vec{n} \in \text{Nul}(A)$,

$$A(\vec{p} + \vec{n}) = A\vec{p} + A\vec{n} = \vec{b} + \vec{0} = \vec{b}$$

so things of form $\vec{p} + \vec{n}$ are solutions.

Now let $\vec{v}$ be any solution, i.e. $A\vec{v} = \vec{b}$

Then $A\vec{v} - A\vec{p} = \vec{0} = A(\vec{v} - \vec{p})$

So $\vec{v} - \vec{p} = \vec{n} \in \text{Nul}(A)$, $\vec{v} = \vec{p} + (\vec{v} - \vec{p}) = \vec{p} + \vec{n}$

Change of Basis

Thm) Let $V \subseteq \mathbb{R}^n$ subspace, $\vec{v}_1, \ldots, \vec{v}_k$ vectors in V. The vectors $\vec{v}_1, \ldots, \vec{v}_k$ form a basis if every $\vec{u} \in V$ can be written as a unique linear combination $\vec{v}_1, \ldots, \vec{v}_k$.

Defn: Let $B = \{v_1, \ldots, v_m\}$ be a basis for V, and let $\vec{x} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$. We call $c_1, \ldots, c_m$ the coordinates of $\vec{x}$ w.r.t. B. We write $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$

Ex: $V = \mathbb{R}^2$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad B = \{\vec{v}_1, \vec{v}_2\}$

1. Show that B is a basis for $\mathbb{R}^2$

Soln: 2 nonparallel lines in $\mathbb{R}^2 \Longrightarrow$ span $\mathbb{R}^2$ ✓

2. If $[\vec{w}]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, what is $\vec{w}$?

Soln. $\vec{w} = 1\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

3. Find B-coords of $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Soln: $\begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$[\vec{v}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$