2/19/25

<u>Recap:</u> <u>Thm</u> (Dimension) Let $V \subseteq \mathbb{R}^n$ be a subspace.

let $\vec{a}_1, \ldots, \vec{a}_k \in V$ be linearly independent and $\vec{b}_1, \ldots, \vec{b}_\ell \in V$ span V. Then $k \leq \ell$

<u>Corollary:</u> If $V \subseteq \mathbb{R}^m$ is a subspace, then any two bases have the same size.

$(k \leq \ell$ and $\ell \leq k \Rightarrow \ell = k)$

<u>Defn:</u> The <u>dimension</u> of a subspace V, written as $\dim V$, is the number of vectors in any basis

<u>Thm</u> Let $V \subseteq \mathbb{R}^m$ be a subspace of dim d.

For $\vec{v}_1, \ldots, \vec{v}_d \in V$, the following are equivalent:

1. $\vec{v}_1, \ldots, \vec{v}_d$ form a basis for V
2. $\vec{v}_1, \ldots, \vec{v}_d$ spans V
3. $\vec{v}_1, \ldots, \vec{v}_d$ are linearly independent

Thm (Rank-Nullity) For any $m \times n$ matrix A, we have:

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = n$$

Pf. $\dim(\mathrm{col}(A)) = \#$ of pivots in A

$\dim(\mathrm{Nul}(A)) = \#$ of free variables

Def^n: For a matrix A, the rank of A is $\dim(\mathrm{col}(A))$

Thm Let $V \subseteq \mathbb{R}^m$ be a subspace. Then every linearly independent set of vectors in V can be extended to a basis.

Ex: Extend $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}$ to a basis of $\mathbb{R}^3$

Soln. ① Find a spanning set of vectors:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$ ("over-doing it" here)

② Find an independent subset with the same span.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

pivots

free variables (dependent vectors)

$\Longrightarrow$ Basis: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Ex: Suppose $\vec{v}_1, \vec{v}_2 \in V$ is a basis.

show that $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$ and $\vec{u}_2 = \vec{v}_1 - \vec{v}_2$ is also a basis.

Soln. $\{\vec{v}_1, \vec{v}_2\}$ basis $\Rightarrow \dim(V) = 2$. So, enough to show that $\vec{u}_1, \vec{u}_2$ span V, i.e.

$v_1, v_2 \in \text{span}\{\vec{u}_1, \vec{u}_2\}$

$\vec{u}_1 = \vec{v}_1 + \vec{v}_2 \qquad \vec{u}_2 = \vec{v}_1 - \vec{v}_2$

$\vec{v}_1 = \vec{u}_1 - \vec{v}_2 \qquad \vec{v}_2 = \vec{v}_1 - \vec{u}_2$

$\vec{v}_1 = \vec{u}_1 - \vec{v}_1 + \vec{u}_2$

$2\vec{v}_1 = \vec{u}_1 + \vec{u}_2$

$\vec{v}_1 = \frac{1}{2}(u_1 + u_2)$

$\vec{v}_2 = \vec{u}_1 - \vec{v}_2 - \vec{u}_2$

$2\vec{v}_2 = \vec{u}_1 - \vec{u}_2$

$\vec{v}_2 = \frac{1}{2}(\vec{u}_1 - \vec{u}_2)$

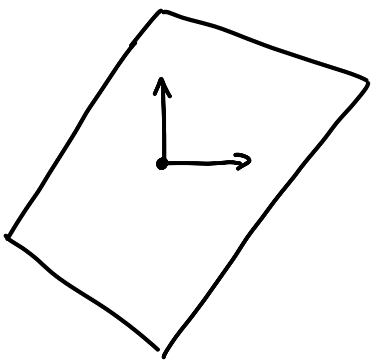
General solution to a linear equation:

Thm Let $A\vec{x} = \vec{b}$ be a consistent system with $\vec{p}$ one solution. Then all solutions have the form $\vec{p} + \vec{n}$ with $\vec{n} \in \mathrm{Nul}(A)$.

Picture:

solutions to $A\vec{x} = \vec{0}$

$\mathrm{Nul}(A)$ is a subspace



Solutions to $A\vec{x} = \vec{b}$ are a translate of the subspace by $\vec{p}$

Think: Solutions to linear systems look linear!